The University of Sydney
Math1003 Integral Calculus and Modelling

Semester 2 **Exercises and Solutions for Week 2** 2009

Assumed Knowledge Sigma notation for sums. The ideas of a sequence of numbers and of the limit of a sequence. Sketching a representative graph of a function given a formula or a set of tabulated values.

Objectives

(1a) To know what is meant by a Riemann sum.

(1b) To understand the definition of the definite integral of a function as the limit of a Riemann sum.

(1c) To understand how to interpret a definite integral of a function in terms of area when the function is not always positive.

(1d) To be able to use the upper and lower sums to obtain an estimate of the error in evaluating a definite integral.

(1e) To be able to find the number of subintervals (sampling frequency) required to reduce the error to a given value in the case of monotonic functions.

Preparatory Questions

1. (*i*) Evaluate the sum $\displaystyle\sum_{k=0}^{4} \frac{(-1)^k}{k+2}$.

(*ii*) Write the following sum using sigma notation: $1 + \dfrac{x}{2} + \dfrac{x^2}{3} + \dfrac{x^3}{4} + \cdots + \dfrac{x^n}{n+1}$.

2. If $c(t)$ represents the cost, in dollars per day, to heat your house, where t is measured in days and $t = 0$ on 1 June 2005, what does $\int_0^{90} c(t)\,dt$ represent?

3. In a chemical reaction, the rate at which a precipitate is formed is a decreasing function of time. In an experiment the following rates were recorded.

time (s)	0	1	2	3	4	5	6
rate (g/s)	12	8.4	5.9	4.1	2.9	2.0	1.4

Sketch the rate as a function of time.

4. Sketch the curve $y = \sqrt{1 + x^3}$.

Practice Questions

5. Using the data in Question 3,

(i) Find an over-estimate, and an under-estimate, for the total mass of precipitate formed in these six seconds.

(ii) How many times a second would measurements have to be made in order to find under-estimates and over-estimates which differ by less than 1g from the exact mass of the precipitate formed in these 6 seconds?

Solution Since we are told that the rate at which the precipitate is formed is a decreasing function of time, the maximum value (M_i) of the rate on the i-th interval occurs at the left endpoint, and the minimum value (m_i) occurs at the right endpoint. Thus we have the following table.

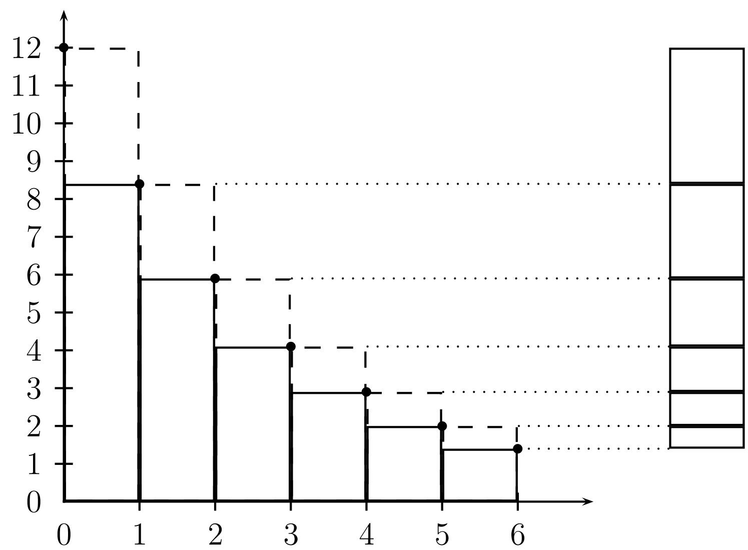
interval i	1	2	3	4	5	6
max M_i	12	8.4	5.9	4.1	2.9	2.0
min m_i	8.4	5.9	4.1	2.9	2.0	1.4

(i) Taking six subintervals, so that $\Delta t = (6-0)/6 = 1$, the over-estimate is given by the upper sum

$$U = \sum_{i=1}^{6} M_i \Delta t = (12 + 8.4 + 5.9 + 4.1 + 2.9 + 2.0) \times 1 = 35.3,$$

and the under-estimate is given by the lower sum

$$L = \sum_{i=1}^{6} m_i \Delta t = 8.4 + 5.9 + 4.1 + 2.9 + 2.0 + 1.4) \times 1 = 24.7.$$



(ii) Looking at the upper and lower sums with 6 subintervals, we find that the difference $U - L$ = area of the rectangle on the right $= (12 - 1.4) \times 1 = 10.6$. If we increase the number of subintervals, the height of the rectangle representing the difference remains unchanged while the width decreases, and so the area (which is $U - L$) decreases also. Since the exact value of the precipitate may lie anywhere between the upper and and lower sums, in order for the over- and under-estimates to lie within 1g of the exact value, we need to make the width of the rectangle small enough so that its area is less than 1. Since the height remains fixed at 10.6, we need the width less than $\frac{1}{10.6}$. So we need to take a measurement at least 11 times every second.

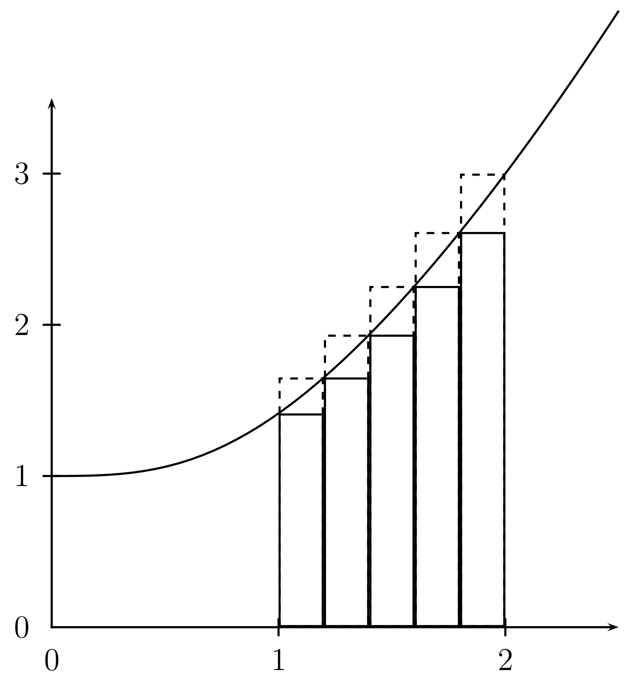
6. Use upper and lower Riemann sums with 5 subintervals to estimate $\int_1^2 \sqrt{1+x^3}\,dx$.

Solution We have

$$L < \int_1^2 \sqrt{1+x^3}\,dx < U,$$

where L is a lower Riemann sum for $\sqrt{1+x^3}$ with $1 \le x \le 2$, and U is an upper Riemann sum.

Now, the function $\sqrt{1+x^3}$ is an increasing function on the interval $[1,2]$, and hence the maximum values occur at the right endpoints, and the minimum values occur at the left endpoints.



With 5 subintervals the width of each subinterval is given by $\Delta x = (2-1)/5 = 0.2$. Now

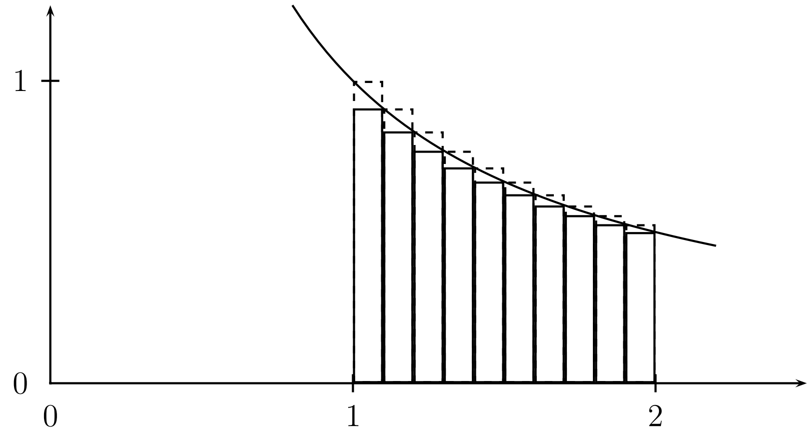
$$U = (\sqrt{1+1.2^3} + \sqrt{1+1.4^3} + \sqrt{1+1.6^3} + \sqrt{1+1.8^3} + \sqrt{1+2^3}) \times 0.2 \approx 2.29,$$
$$L = (\sqrt{1+1^3} + \sqrt{1+1.2^3} + \sqrt{1+1.4^3} + \sqrt{1+1.6^3} + \sqrt{1+1.8^3}) \times 0.2 \approx 1.97\,.$$

The exact value of the integral lies somewhere between L and U. That is,

$$1.97 < \int_1^2 \sqrt{1+x^3}\,dx < 2.29.$$

7. Draw diagrams illustrating the approximation of $\ln 2 = \int_1^2 \frac{dt}{t}$ using upper and lower Riemann sums with 10 subdivisions.
Estimate $\ln 2$ correct to one decimal place by calculating the upper and lower Riemann sums.

Solution



Then

$$\begin{aligned} \text{the lower sum } &= \frac{1}{10}\left(\frac{1}{1.1}+\frac{1}{1.2}+\cdots+\frac{1}{2.0}\right) = 0.66877\ldots \\ \text{and the upper sum } &= \frac{1}{10}\left(\frac{1}{1.0}+\frac{1}{1.1}+\cdots+\frac{1}{1.9}\right) = 0.71877\ldots. \end{aligned}$$

So $\ln 2 = 0.7$ correct to 1 decimal place.

More Questions

8. A speed-boat accelerates from rest (with increasing velocity), reaching a speed of 40 m/sec as it moves in a straight line over a period of 20 seconds. Velocities were measured every 2 seconds, and recorded in the following table:

time (sec)	0	2	4	6	8	10	12	14	16	18	20
vel (m/sec)	0	3	8	13	19	25	29	33	36	38	40

(i) Use lower and upper Riemann sums to find lower and upper bounds for the distance travelled by the boat.

(ii) How often would measurements need to be taken to guarantee that lower and upper Riemann sums differ from the actual distance travelled by less than 10 m?

Solution

(i) The lower sum L is found by using the minimum velocity over each 2-second interval. So

$$L = 2(0+3+8+13+19+25+29+33+36+38) = 408.$$

The upper sum U is found by using the maximum velocity over each 2-second interval. So

$$U = 2(3+8+13+19+25+29+33+36+38+40) = 488.$$

$$\text{Hence, } 408\text{m} \le \text{distance travelled} \le 488\text{m}.$$

(ii) Suppose we divide the 20 second interval into N intervals of equal length $20/N$ seconds. Then the the difference between the upper and lower estimates of the distance travelled is $40 \times 20/N = 800/N$ metres. For the estimate to be less than 10 m, we require $800/N < 10$ or equivalently $N > 80$, i.e $N \geq 81$. So velocities would have to be measured every $20/81$ ths of a second.

If we are satisfied with less than or equal to 10 m then velocities only have to measured every $20/80$ ths $= 1/4$ of a second, i.e. 4 times a second.

If you set out to find how many times a second measurements would have to be made to make the difference between the upper and lower Riemann sum estimates less than 10 m then you should have come up with the answer 5.

Answers to Preparatory Questions

1. (i) $\displaystyle\sum_{k=0}^{4} \frac{(-1)^k}{k+2} = \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} = \frac{23}{60}.$

(ii) $\displaystyle 1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \cdots + \frac{x^n}{n+1} = \sum_{k=0}^{n} \frac{x^k}{k+1}.$

2. $\int_0^{90} c(t)\,dt$ represents the heating cost, in dollars, for 90 days from 1 June 2005, or roughly for the months June, July, and August.