The University of Sydney
Math1003 Integral Calculus and Modelling

Semester 2 | **Exercises for Week 3** | 2009

Assumed Knowledge Integrals of simple functions such as x^n (including $1/x$), $\sin x$, $\cos x$, e^x. The simple properties of definite integrals.

Objectives

(2a) To be able to estimate the definite integral of a function and the error of the estimate using Riemann sums.

(2b) To be able to estimate a sum of a series by comparing it with a definite integral.

(2c) To understand that integration and differentiation are inverse operations, i.e. that the definite integral of the derivative of a function is equal to the difference between the function values at the end-points, $\int_a^b F'(x)\,dx = F(b) - F(a)$.

(2d) To understand the conditions under which the definite integral of a function exists.

(2e) To be able to use the properties of definite integrals confidently.

Preparatory Questions

1. Let f and g be continuous functions on the interval $[a, b]$, and let $a \le c \le b$.

(i) If $\int_a^b f(x)dx = 10$ and $\int_c^b 2f(x)dx = 5$, find $\int_a^c f(x)dx$.

(ii) If $\int_a^b f(x)dx = 12$ and $\int_a^b (2f(x) + 3g(x))dx = 63$, find $\int_a^b g(x)dx$.

(iii) If $\int_c^a f(x)dx = 5$ and $\int_b^c f(x)dx = -2$, find $\int_a^b 2f(x)dx$.

2. A MATH1003 student reasons that because $-1/x$ has derivative $1/x^2$ the Fundamental Theorem of Calculus gives:

$$\int_{-1}^{1} \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_{-1}^{1} = -2.$$

Her friend says: "But $\frac{1}{x^2}$ is always positive, so its graph is above the x-axis. So surely, the integral is a positive number."
Which argument do you find more persuasive?

Practice Questions

3. Sort out the apparent contradiction in Question 2.

4. (*i*) Use the Fundamental Theorem to find $\int_1^a \frac{1}{x^2}\,dx$ (where a is any real number greater than 1).

(*ii*) What is $\lim_{a\to\infty} \int_1^a \frac{1}{x^2}\,dx$?

(*iii*) Draw a sketch of the function $\frac{1}{x^2}$ and use it to interpret your answer to part (*ii*) geometrically.

(*iv*) Find a lower Riemann sum for the function $\frac{1}{x^2}$ on the interval $[1, \infty)$, using sub-intervals of width 1.

(*v*) Use parts (*ii*) and (*iv*) to show that $\sum_{n=1}^{\infty} \frac{1}{n^2} < 2$.

(*vi*) Calculate $\sum_{n=1}^{7} \frac{1}{n^2}$. (*vii*) Deduce that $1.5 < \sum_{n=1}^{\infty} \frac{1}{n^2} < 2$.

5. (Suitable for group work and discussion) Use an appropriate integral, and Riemann sums, to find an estimate for $1 + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{100}$.

More Questions

6. Consider a partition of $[0, 1.75]$ into 7 equal parts. Given the values of $\cos(x^2)$ tabulated below, use lower and upper Riemann sums to find lower and upper estimates for $\int_0^{1.75} \cos(x^2)\,dx$.

x	0	0.25	0.5	0.75	1	1.25	1.5	1.75
$\cos(x^2)$	1	0.998	0.969	0.846	0.54	0.008	−0.628	−0.997

Remember that x should be measured in radians.

7. (*i*) Consider the partition of $[0, 1]$ into five equal parts. Use lower and upper Riemann sums to find lower and upper bounds for $\int_0^1 x^5\,dx$.

(*ii*) Obtain lower and upper bounds for $\sum_{i=1}^{25} i^4$.

Answers to Selected Questions

1. (*i*) 7.5 (*ii*) 13 (*iii*) −6

4. (*i*) $-\frac{1}{a} + 1$ (*ii*) 1 (*iv*) $\sum_{n=2}^{\infty} \frac{1}{n^2}$ (*vi*) 1.511 . . .

5. $\frac{2000}{3} < 1 + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{99} + \sqrt{100} < \frac{2030}{3}$.

6. $0.434 \le \int_0^{1.75} \cos(x^2)\,dx \le 0.93325$.

7. (*i*) $0.0832 \le \int_0^1 x^5\,dx \le 0.2832$. (*ii*) $1953125 \le \sum_{i=1}^{25} i^4 \le 2343750$.