Semester 2 **Exercises and Solutions for Week 3** 2009

Assumed Knowledge Integrals of simple functions such as x^n (including $1/x$), $\sin x$, $\cos x$, e^x. The simple properties of definite integrals.

Objectives

(2a) To be able to estimate the definite integral of a function and the error of the estimate using Riemann sums.

(2b) To be able to estimate a sum of a series by comparing it with a definite integral.

(2c) To understand that integration and differentiation are inverse operations, i.e. that the definite integral of the derivative of a function is equal to the difference between the function values at the end-points, $\int_a^b F'(x)\,dx = F(b) - F(a)$.

(2d) To understand the conditions under which the definite integral of a function exists.

(2e) To be able to use the properties of definite integrals confidently.

Preparatory Questions

1. Let f and g be continuous functions on the interval $[a, b]$, and let $a \le c \le b$.

(*i*) If $\int_a^b f(x)dx = 10$ and $\int_c^b 2f(x)dx = 5$, find $\int_a^c f(x)dx$.

(*ii*) If $\int_a^b f(x)dx = 12$ and $\int_a^b (2f(x) + 3g(x))dx = 63$, find $\int_a^b g(x)dx$.

(*iii*) If $\int_c^a f(x)dx = 5$ and $\int_b^c f(x)dx = -2$, find $\int_a^b 2f(x)dx$.

2. A MATH1003 student reasons that because $-1/x$ has derivative $1/x^2$ the Fundamental Theorem of Calculus gives:

$$\int_{-1}^{1} \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_{-1}^{1} = -2.$$

Her friend says: "But $\dfrac{1}{x^2}$ is always positive, so its graph is above the x-axis. So surely, the integral is a positive number."
Which argument do you find more persuasive?

Practice Questions

3. Sort out the apparent contradiction in Question 2.

Solution

The function $f(x) = \dfrac{1}{x^2}$ is not continuous on $[-1, 1]$ as it is undefined at $x = 0$. So the Fundamental Theorem cannot be applied to it on the interval $[-1, 1]$, as the first student did.

The second student has overlooked this discontinuity too, when they claimed the integrand is positive on $[-1, 1]$, as techincially this statement does not make sense at $x = 0$. The discontinuity of the integrand at $x = 0$ means that the student cannot sure the integral has a numeric value, and in fact this integral does not have a numeric value.

4. (*i*) Use the Fundamental Theorem to find $\displaystyle\int_1^a \frac{1}{x^2}\,dx$ (where a is any real number greater than 1).

(*ii*) What is $\displaystyle\lim_{a\to\infty}\int_1^a \frac{1}{x^2}\,dx$?

(*iii*) Draw a sketch of the function $\dfrac{1}{x^2}$ and use it to interpret your answer to part (*ii*) geometrically.

(*iv*) Find a lower Riemann sum for the function $\dfrac{1}{x^2}$ on the interval $[1, \infty)$, using sub-intervals of width 1.

(*v*) Use parts (*ii*) and (*iv*) to show that $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2} < 2$.

(*vi*) Calculate $\displaystyle\sum_{n=1}^{7}\frac{1}{n^2}$.

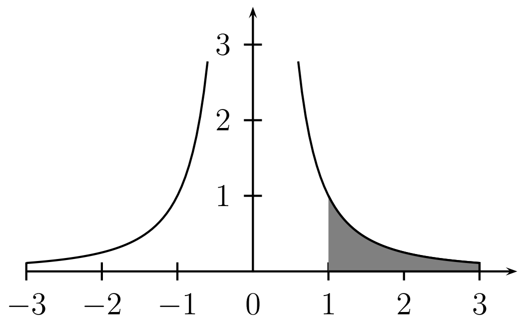
(*vii*) Deduce that $\displaystyle 1.5 < \sum_{n=1}^{\infty}\frac{1}{n^2} < 2$.

Solution

(*i*) $\displaystyle\int_1^a \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^a = -\frac{1}{a} + 1.$

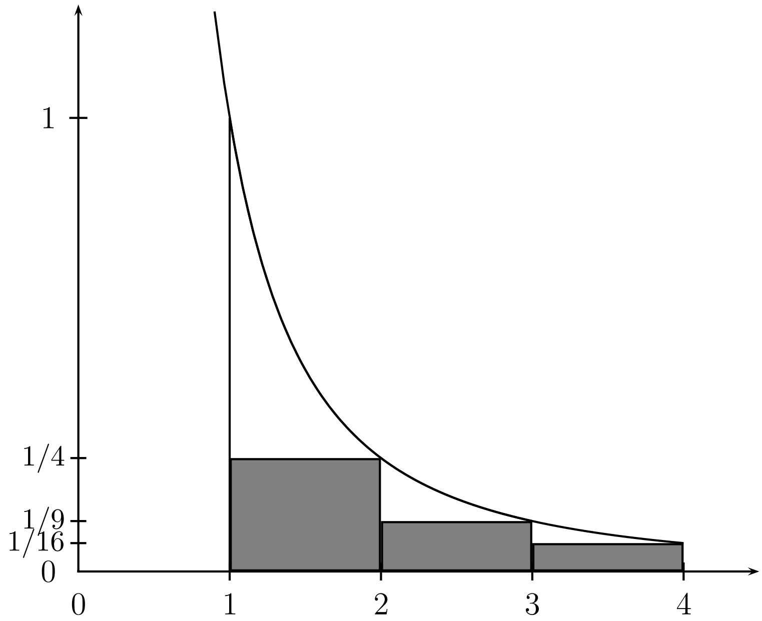
(*ii*) As $a \to \infty$, $\dfrac{1}{a} \to 0$, and so $\displaystyle\lim_{a\to\infty}\int_1^a \frac{1}{x^2}\,dx = 1.$

(*iii*) The area bounded by the x-axis, the graph of $y = \dfrac{1}{x^2}$, and the line $x = 1$, for $x \geq 1$, is equal to 1 square unit. (This area is shaded in the diagram.)



(*iv*) As we can see from the diagram, a lower Riemann sum is

$$1 \times \left(\frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots\right) = \sum_{n=2}^{\infty}\frac{1}{n^2}.$$



(*v*) The Riemann sum in part (*iv*) is less than the integral calculated in part (*ii*). So we have $\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots<1$, and adding 1 to both sides of this inequality gives $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots<2$. That is, $\sum_{n=1}^{\infty}\frac{1}{n^2}<2$.

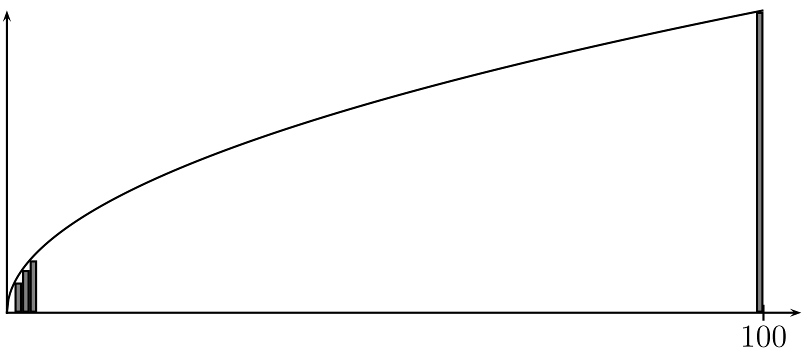
(*vi*) $\sum_{n=1}^{7}\frac{1}{n^2}=1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\frac{1}{36}+\frac{1}{49}=1.511\ldots.$

(*vii*) Clearly, $\sum_{n=1}^{\infty}\frac{1}{n^2}>\sum_{n=1}^{7}\frac{1}{n^2}$, and so we have $1.5<\sum_{n=1}^{\infty}\frac{1}{n^2}<2$.

(In fact, $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$; a most remarkable formula, as I'm sure you would agree. If you study mathematics in second year, you will find out why this is so!)

5. (Suitable for group work and discussion) Use an appropriate integral, and Riemann sums, to find an estimate for $1+\sqrt{2}+\sqrt{3}+\cdots+\sqrt{100}$.

Solution Consider the function $f(x)=\sqrt{x}$ on the interval [0, 100].



Let L = the lower Riemann sum formed by taking sub-intervals of width 1, and U = the upper Riemann sum formed by taking sub-intervals of width 1. Then

$$L=0+1+\sqrt{2}+\sqrt{3}+\cdots+\sqrt{99},$$

$$U = 1 + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{100}.$$

Since $L < \int_0^{100} \sqrt{x}\,dx < U$, and $\int_0^{100} \sqrt{x}\,dx = \left[\frac{2}{3}x^{3/2}\right]_0^{100} = \frac{2000}{3}$, we have

$$1 + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{99} < \frac{2000}{3} < 1 + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{100}.$$

Taking the left-hand inequality, and adding $\sqrt{100}$ gives

$$1 + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{99} + \sqrt{100} < \frac{2000}{3} + 10 = \frac{2030}{3}.$$

So we have $\frac{2000}{3} < 1 + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{99} + \sqrt{100} < \frac{2030}{3}$.

More Questions

6. Consider a partition of $[0, 1.75]$ into 7 equal parts. Given the values of $\cos(x^2)$ tabulated below, use lower and upper Riemann sums to find lower and upper estimates for $\int_0^{1.75} \cos(x^2)\,dx$.

x	0	0.25	0.5	0.75	1	1.25	1.5	1.75
$\cos(x^2)$	1	0.998	0.969	0.846	0.54	0.008	−0.628	−0.997

Remember that x should be measured in radians.

Solution

$$L_7 = (0.998 + 0.969 + 0.846 + 0.54 + 0.008 - 0.628 - 0.997) \times 0.25 = 1.736 \times 0.25 = 0.434$$
$$U_7 = (1 + 0.998 + 0.969 + 0.846 + 0.54 + 0.008 - 0.628) \times 0.25 = 3.733 \times 0.25 = 0.93325$$

so

$$0.434 \le \int_0^{1.75} \cos(x^2)\,dx \le 0.93325\,.$$

7. (*i*) Consider the partition of $[0, 1]$ into five equal parts. Use lower and upper Riemann sums to find lower and upper bounds for $\int_0^1 x^5\,dx$.

(*ii*) Obtain lower and upper bounds for $\sum_{i=1}^{25} i^4$.

Solution

(*i*)

$$U_5 = \left((0.2)^5 + (0.4)^5 + (0.6)^5 + (0.8)^5 + 1^5\right) \times 0.2 = 1.416 \times 0.2 = 0.2832$$
$$L_5 = \left(0^5 + (0.2)^5 + (0.4)^5 + (0.6)^5 + (0.8)^5\right) \times 0.2 = 0.416 \times 0.2 = 0.0832$$

so

$$0.0832 \le \int_0^1 x^5\,dx \le 0.2832\,.$$

(*ii*) Choose an interval $[0,25]$ and partition it into 25 subintervals of width 1. Then the lower and upper sums for $\int_0^{25} x^4\,dx$ are

$$L_{25} = \left(0^4 + 1^4 + 2^4 \ldots 24^4\right) \times 1 = \sum_{i=1}^{24} i^4$$
$$U_{25} = \left(1^4 + 2^4 + 3^4 \ldots 25^4\right) \times 1 = \sum_{i=1}^{25} i^4.$$

Hence $\sum_{i=1}^{24} i^4 \le \int_0^{25} x^4\,dx \le \sum_{i=1}^{25} i^4$.

Using the Fundamental Theorem, $\int_0^{25} x^4\,dx = \left[\frac{x^5}{5}\right]_0^{25} = \frac{(25)^5}{5} = 1953125$,

so $1953125 \le \sum_{i=1}^{25} i^4$ and $\sum_{i=1}^{24} i^4 \le 1953125$. Adding the extra term $(25)^4$ to both sides of the latter inequality we find that $\sum_{i=1}^{25} i^4 \le 1953125 + (25)^4 = 2343750$.

Hence

$$1953125 \le \sum_{i=1}^{25} i^4 \le 2343750\,.$$

The actual value of the sum is 2153645.

Answers to Selected Questions

1. (*i*) 7.5 (*ii*) 13 (*iii*) -6

Solution

(*i*) Since $\int_b^c f(x)dx = -5/2$, $\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx = 10 - 2.5 = 7.5$.

(*ii*) $\int_a^b (2f(x) + 3g(x))dx = 2\int_a^b f(x)dx + 3\int_a^b g(x)dx = 24 + 3\int_a^b g(x)dx = 63$. Therefore, $\int_a^b g(x)dx = (63-24)/3 = 13$.

(*iii*)

$$\int_a^b 2f(x)dx = 2\left(\int_a^c f(x)dx + \int_c^b f(x)dx\right) = 2\left(-\int_c^a f(x)dx - \int_b^c f(x)dx\right)$$
$$= 2(-5+2) = -6.$$

4. (*i*) $-\frac{1}{a} + 1$ (*ii*) 1 (*iv*) $\sum_{n=2}^{\infty} \frac{1}{n^2}$ (*vi*) $1.511\ldots$

5. $\frac{2000}{3} < 1 + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{99} + \sqrt{100} < \frac{2030}{3}$.

6. $0.434 \le \int_0^{1.75} \cos(x^2)\,dx \le 0.93325\,.$

7. (*i*) $0.0832 \le \int_0^1 x^5\,dx \le 0.2832\,.$ (*ii*) $1953125 \le \sum_{i=1}^{25} i^4 \le 2343750\,.$