

Assumed Knowledge Sketching curves of simple functions. Integrals of simple functions such as x^n (including $1/x$), $\sin x$, $\cos x$, e^x .

Objectives

- (3a) To be able to interpret an area or volume as the limit of an appropriate Riemann sum.
- (3b) To be able to use elementary slices to construct Riemann sums for areas and volumes.
- (3c) To be able to use elementary shells to construct Riemann sums for volumes.
- (3d) To understand the substitution formula and be able to use simple substitutions to evaluate definite integrals.

Preparatory Questions

1. Evaluate the following definite integrals:

$$(i) \int_0^{5\pi} \cos\left(\frac{x}{10}\right) dx. \quad (ii) \int_1^2 \sqrt{x-1} dx. \quad (iii) \int_0^1 x^\pi dx.$$

2. Sketch the region bounded by the curves $y = \sin x$ and $y = \sin 2x$, and the straight lines $x = 0$ and $x = \pi/2$.
3. Sketch the region of the xy -plane bounded by the x -axis, the line $x = 2$, and the graph of $y = x$. This region is rotated about the line $x = 4$. Sketch an elementary disk that you would use to construct the volume.

Practice Questions

4. Evaluate the following definite integrals by making a substitution.

$$(i) \int_0^1 \frac{x^2}{\sqrt{2+x^3}} dx. \quad (ii) \int_0^1 (2x+1)(x^2+x+1)^3 dx.$$
$$(iii) \int_0^{\pi/2} \cos^3 x dx. \text{ Hint: First use the identity } \cos^2 x = 1 - \sin^2 x.$$

5. Find the area of the region sketched in Question 2.
6. Calculate the volume generated in Question 3.
7. Suppose that a bagel cut horizontally in half has the shape given by rotating, about the y axis, the area bounded by the curve $y = 3x - x^2 - 2$ and the x -axis.
- (i) Sketch the curve $y = 3x - x^2 - 2$.
 - (ii) Consider a vertical strip under the graph at some point x ($1 \leq x \leq 2$), of width Δx , which is rotated about the y axis to give a cylindrical shell. Sketch the shell.
 - (iii) Imagine the shell being cut vertically and opened out flat. Thus find the volume of the cylindrical shell.

- (iv) Write down the volume of the half bagel as a definite integral.
 (v) Evaluate the definite integral to find the volume of the half bagel.

More Questions

8. Evaluate the following definite integrals using the substitution $u = \sec x$.

(i) $\int_0^{\pi/4} \tan x \sec^3 x \, dx.$ (ii) $\int_0^{\pi/3} \sec^5 x \tan^3 x \, dx.$

Hint: Use a trigonometric identity in (ii).

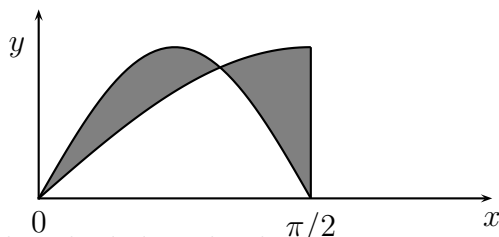
9. Evaluate the following definite integrals using a substitution.

(i) $\int_0^{1/2} \frac{x}{\sqrt{1-x^2}} \, dx.$ (ii) $\int_0^{\pi/2} \sin^3 x \cos^4 x \, dx.$

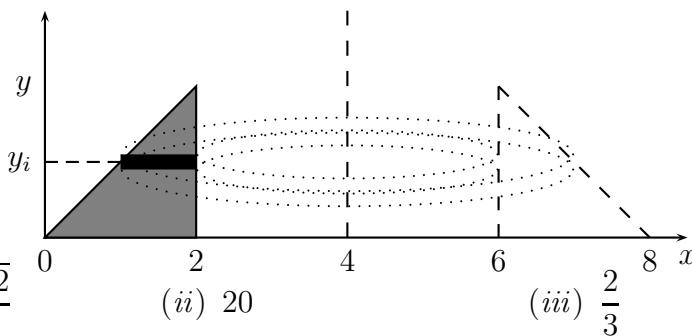
Answers to Selected Questions

1. (i) 10. (ii) $\frac{2}{3}.$ (iii) $\frac{1}{\pi+1}.$

2. The curves intersect when $\sin x = \sin 2x$, i.e., when $\sin x = 2 \sin x \cos x$ or $\sin x(1 - 2 \cos x) = 0$. The roots occur when $\sin x = 0$ and $\cos x = 1/2$. For $0 \leq x \leq \pi/2$, $\sin x = 0$ when $x = 0$ and $\cos x = 1/2$ when $x = \pi/3$.



3. The region to be rotated is shaded in the diagram.



4. (i) $\frac{2\sqrt{3}}{3} - \frac{2\sqrt{2}}{3}$ (ii) 20 (iii) $\frac{2}{3}$

5. $\frac{1}{2}$

6. $\frac{32\pi}{3}$

7. (iii) $2\pi(3x^2 - x^3 - 2x)\Delta x$ (iv) $\int_1^2 2\pi(3x^2 - x^3 - 2x) \, dx$ (v) $\frac{\pi}{2}$

8. (i) $\frac{2\sqrt{2}}{3} - \frac{1}{3}$ (ii) $\frac{418}{35}$

9. (i) $-\frac{\sqrt{3}}{2} + 1$ (ii) $\frac{2}{35}$