The University of Sydney
Math1003 Integral Calculus and Modelling

Semester 2 **Exercises and Solutions for Week 4** 2009

Assumed Knowledge Sketching curves of simple functions. Integrals of simple functions such as x^n (including $1/x$), $\sin x$, $\cos x$, e^x.

Objectives

(3a) To be able to interpret an area or volume as the limit of an appropriate Riemann sum.

(3b) To be able to use elementary slices to construct Riemann sums for areas and volumes.

(3c) To be able to use elementary shells to construct Riemann sums for volumes.

(3d) To understand the substitution formula and be able to use simple substitutions to evaluate definite integrals.

Preparatory Questions

1. Evaluate the following definite integrals:

$$(i)\ \int_0^{5\pi} \cos\left(\frac{x}{10}\right) dx. \qquad (ii)\ \int_1^2 \sqrt{x-1}\, dx. \qquad (iii)\ \int_0^1 x^\pi\, dx.$$

2. Sketch the region bounded by the curves $y = \sin x$ and $y = \sin 2x$, and the straight lines $x = 0$ and $x = \pi/2$.

3. Sketch the region of the xy-plane bounded by the x-axis, the line $x = 2$, and the graph of $y = x$. This region is rotated about the line $x = 4$. Sketch an elementary disk that you would use to construct the volume.

Practice Questions

4. Evaluate the following definite integrals by making a substitution.

$$(i)\ \int_0^1 \frac{x^2}{\sqrt{2+x^3}} dx. \qquad (ii)\ \int_0^1 (2x+1)(x^2+x+1)^3 dx.$$

(iii) $\displaystyle\int_0^{\pi/2} \cos^3 x\, dx$. Hint: First use the identity $\cos^2 x = 1 - \sin^2 x$.

Solution

(i) Use the substitution $u = 2 + x^3$; then $du = 3x^2 dx$, $u(0) = 2$, $u(1) = 3$ and so

$$\int_0^1 \frac{x^2}{\sqrt{2+x^3}} dx = \int_2^3 \frac{\frac{1}{3}du}{\sqrt{u}} = \left[\frac{2}{3}\sqrt{u}\right]_2^3 = \frac{2\sqrt{3}}{3} - \frac{2\sqrt{2}}{3}.$$

(ii) Use the substitution $u = x^2 + x + 1$; then $du = (2x+1)dx$, $u(0) = 1$, $u(1) = 3$ and so

$$\int_0^1 (2x+1)(x^2+x+1)^3 dx = \int_1^3 u^3 du = \left[\frac{u^4}{4}\right]_1^3 = \frac{81}{4} - \frac{1}{4} = 20\,.$$

(*iii*) Use the identity $\cos^2 x = 1 - \sin^2 x$ and the substitution $u = \sin x$; then $du = \cos x\, dx$, $u(0) = 0$, $u(\pi/2) = 1$ and so

$$\int_0^{\pi/2} \cos^3 x\, dx = \int_0^{\pi/2} (1 - \sin^2 x) \cos x\, dx = \int_0^1 (1 - u^2) du = \left[u - \frac{u^3}{3}\right]_0^1 = \frac{2}{3}.$$

5. Find the area of the region sketched in Question 2.

Solution The interval must be split into two at $x = \pi/3$ and the (positive) area of each region added. So the area of the whole region is given by

$$\int_0^{\pi/3} (\sin 2x - \sin x)\, dx + \int_{\pi/3}^{\pi/2} (\sin x - \sin 2x)\, dx$$

$$= \left[-\frac{\cos 2x}{2} + \cos x\right]_0^{\pi/3} + \left[-\cos x + \frac{\cos 2x}{2}\right]_{\pi/3}^{\pi/2}$$

$$= \frac{1}{4} + \frac{1}{2} + \frac{1}{2} - 1 + 0 - \frac{1}{2} + \frac{1}{2} + \frac{1}{4} = \frac{1}{2}.$$

6. Calculate the volume generated in Question 3.

Solution Take a small horizontal slice of the region at some point y_i (between 0 and 2), of width Δy. This is the black rectangle in the diagram. Rotating this slice about the line $x = 4$ gives a disc with a hole, as illustrated. Writing R for the outside radius, and r for the radius of the hole, we have the volume of the disk with a hole $\Delta V \approx \pi(R^2 - r^2)\Delta y$, where $R = 4 - y_i$ and $r = 2$.
That is, $\Delta V \approx \pi\big((4 - y_i)^2 - (2)^2\big)\Delta y = \pi(12 - 8y_i + y_i^2)\Delta y$.
The required volume is therefore given by the definite integral

$$\int_0^2 \pi(12 - 8y + y^2)\, dy = \pi\left[12y - 4y^2 + \frac{y^3}{3}\right]_0^2 = \pi\left(24 - 16 + \frac{8}{3}\right) = \frac{32\pi}{3}.$$

7. Suppose that a bagel cut horizontally in half has the shape given by rotating, about the y axis, the area bounded by the curve $y = 3x - x^2 - 2$ and the x-axis.

(*i*) Sketch the curve $y = 3x - x^2 - 2$.

(*ii*) Consider a vertical strip under the graph at some point x ($1 \le x \le 2$), of width Δx, which is rotated about the y axis to give a cylindrical shell. Sketch the shell.

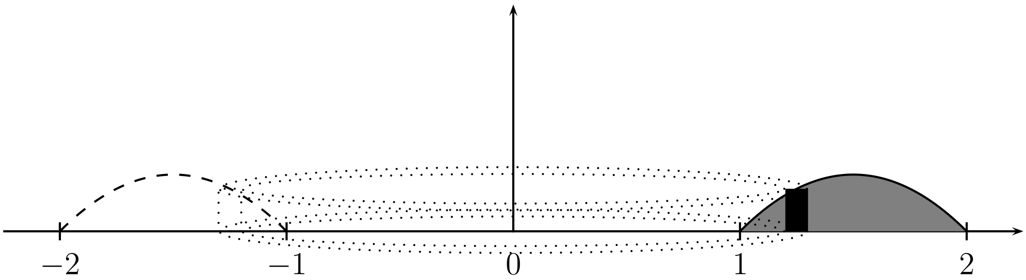
(*iii*) Imagine the shell being cut vertically and opened out flat. Thus find the volume of the cylindrical shell.

(*iv*) Write down the volume of the half bagel as a definite integral.

(*v*) Evaluate the definite integral to find the volume of the half bagel.

Solution

(*i*) The bagel has the following form:



(*ii*) The elementary shell is also shown in the diagram.

(*iii*) The cylindrical shell has radius x, and height $(3x - x^2 - 2)$. When we make the vertical cut and open it out, we obtain an object which approximates a rectangular slab of length $2\pi x$ and height $(3x - x^2 - 2)$.
The area of this slab is $\Delta A = 2\pi x \times (3x - x^2 - 2) = 2\pi(3x^2 - x^3 - 2x)$, and the volume is $\Delta V = \Delta A \times \Delta x = 2\pi(3x^2 - x^3 - 2x)\Delta x$.

(*iv*) The volume of the half bagel, as a definite integral, is

$$V = \int_1^2 2\pi(3x^2 - x^3 - 2x)dx\,.$$

(*v*) $V = 2\pi\left[x^3 - \frac{1}{4}x^4 - x^2\right]_1^2 = 2\pi\left\{(8-4-4) - \left(1 - \frac{1}{4} - 1\right)\right\} = 2\pi \times \frac{1}{4} = \dfrac{\pi}{2}$.

More Questions

8. Evaluate the following definite integrals using the substitution $u = \sec x$.

(*i*) $\displaystyle\int_0^{\pi/4} \tan x \sec^3 x\, dx$. (*ii*) $\displaystyle\int_0^{\pi/3} \sec^5 x \tan^3 x\, dx$.

Hint: Use a trigonometric identity in (*ii*).

Solution

(*i*) Using the substitution $u = \sec x$, $du = \tan x \sec x\, dx$, $u(0) = 1$, $u(\pi/4) = \sqrt{2}$ and so

$$\int_0^{\pi/4} \tan x \sec^3 x\, dx = \int_0^{\pi/4} \sec^2(\sec x \tan x)dx = \int_1^{\sqrt{2}} u^2 du = \left[\frac{u^3}{3}\right]_1^{\sqrt{2}} = \frac{2\sqrt{2}}{3} - \frac{1}{3}\,.$$

(*ii*) Using the same substitution and the identity $\tan^2 x = \sec^2 x - 1$, $du = \tan x \sec x$ (as before), $u(0) = 1$, $u(\pi/3) = 2$ and so

$$\begin{aligned}\int_0^{\pi/3} \sec^5 x \tan^3 x\, dx &= \int_0^{\pi/3} \sec^4 x \tan^2 x \sec x \tan x\, dx \\ &= \int_0^{\pi/3} \sec^4 x(\sec^2 x - 1) \sec x \tan x\, dx \\ &= \int_1^2 (u^6 - u^4)du = \left[\frac{u^7}{7} - \frac{u^5}{5}\right]_1^2 = \frac{127}{7} - \frac{31}{5} = \frac{418}{35}\,.\end{aligned}$$

9. Evaluate the following definite integrals using a substitution.

(i) $\displaystyle\int_0^{1/2} \frac{x}{\sqrt{1-x^2}}dx.$ $\qquad$ (ii) $\displaystyle\int_0^{\pi/2} \sin^3 x \cos^4 x \, dx.$

Solution

(i) Use the substitution $u = 1 - x^2$, $du = -2x$, $u(0) = 1$, $u(1/2) = 3/4$ so that

$$\int_0^{1/2} \frac{x}{\sqrt{1-x^2}}dx = \int_1^{3/4} \frac{-\frac{1}{2}du}{\sqrt{u}} = \left[-\sqrt{u}\right]_1^{3/4} = -\frac{\sqrt{3}}{2} + 1\,.$$

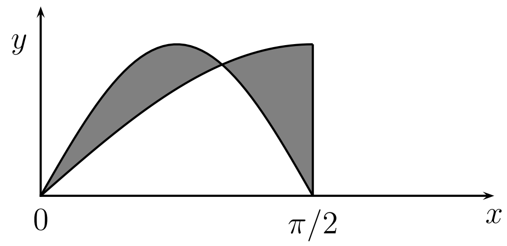
(ii) First use the identity $\sin^2 x = 1 - \cos^2 x$ and then the substitution $u = \cos x$, $du = -\sin x \, dx$, $u(0) = 1$, $u(\pi/2) = 0$ so that

$$\begin{aligned}\int_0^{\pi/2} \sin^3 x \cos^4 x \, dx &= \int_0^{\pi/2} \sin x (1 - \cos^2 x) \cos^4 x \, dx \\ &= \int_0^{\pi/2} (\cos^4 x - \cos^6 x) \sin x \, dx \\ &= \int_1^0 -(u^4 - u^6) \, du \\ &= \left[-\frac{u^5}{5} + \frac{u^7}{7}\right]_1^0 = \frac{1}{5} - \frac{1}{7} = \frac{2}{35}\,.\end{aligned}$$

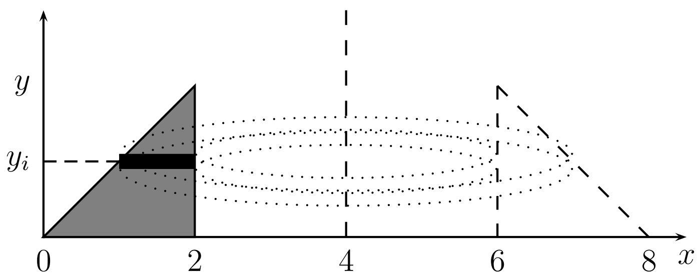
Answers to Selected Questions

1. (i) 10. $\qquad$ (ii) $\frac{2}{3}$. $\qquad$ (iii) $\frac{1}{\pi+1}$.

2. The curves intersect when $\sin x = \sin 2x$, i.e., when $\sin x = 2\sin x \cos x$ or $\sin x(1 - 2\cos x) = 0$. The roots occur when $\sin x = 0$ and $\cos x = 1/2$. For $0 \le x \le \pi/2$, $\sin x = 0$ when $x = 0$ and $\cos x = 1/2$ when $x = \pi/3$.



3. The region to be rotated is shaded in the diagram.



4. (i) $\frac{2\sqrt{3}}{3} - \frac{2\sqrt{2}}{3}$ $\qquad$ (ii) 20 $\qquad$ (iii) $\frac{2}{3}$

5. $\frac{1}{2}$

6. $\dfrac{32\pi}{3}$

7. (iii) $2\pi(3x^2 - x^3 - 2x)\Delta x$ (iv) $\displaystyle\int_1^2 2\pi(3x^2 - x^3 - 2x)dx$ (v) $\dfrac{\pi}{2}$

8. (i) $\dfrac{2\sqrt{2}}{3} - \dfrac{1}{3}$ (ii) $\dfrac{418}{35}$

9. (i) $-\dfrac{\sqrt{3}}{2} + 1$ (ii) $\dfrac{2}{35}$