

**Assumed Knowledge** Sketching curves of simple functions. Integrals of simple functions such as  $x^n$  (including  $1/x$ ),  $\sin x$ ,  $\cos x$ ,  $e^x$ .

### Objectives

- (4a) To understand and be able to use integration by parts to evaluate definite integrals.
- (4b) To understand that an indefinite integral is a function.
- (4c) To understand that differentiation and (indefinite) integration are inverse processes when applied to functions.
- (4d) To be able to sketch a function defined by an integral.
- (4e) To be able to derive a reduction formula for an integral.

### Preparatory Questions

1. (i) Find the indefinite integral  $\int \tan x \, dx$ .

Hint: Use  $\tan x = \frac{\sin x}{\cos x}$  and substitute  $u = \cos x$ .

- (ii) Use two integration by part to find the indefinite integral  $\int x^2 e^x \, dx$ .

### Practice Questions

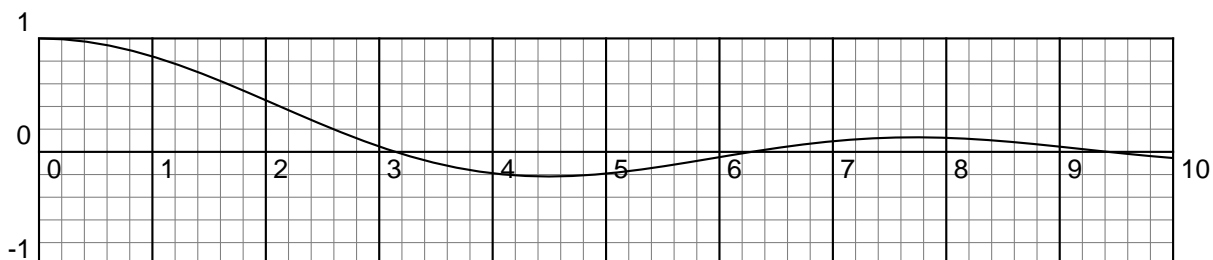
2. Evaluate the following integrals by using integration by parts.

(i)  $\int_0^{1/2} x e^{2x} \, dx$ .      (ii)  $\int_0^{\pi/4} \theta \sin 4\theta \, d\theta$ .      (iii)  $\int_1^2 t^2 \ln t \, dt$ .

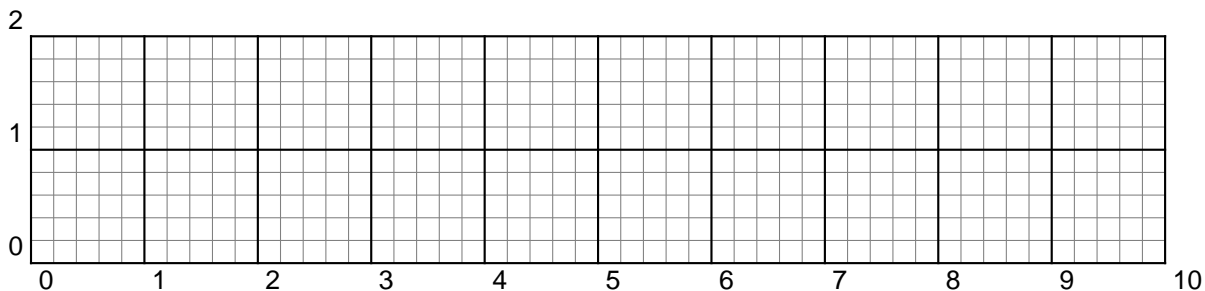
3. Define  $\text{Si}(x)$  as  $\text{Si}(x) = \int_0^x f(t) \, dt$ , where  $f(t) = \begin{cases} \frac{\sin t}{t} & t \neq 0 \\ 1 & t = 0. \end{cases}$

This function is called the *sine-integral*, and is useful in optics.

This is the graph of  $f(t)$ .



- (i) What is  $\text{Si}'(x)$ ? (ii) What is the value of  $\text{Si}(0)$ ?
- (iii) For  $0 \leq x \leq 3\pi$ , use the graph of  $f(t)$  to determine the values of  $x$  for which  $\text{Si}(x)$  is increasing, and the values of  $x$  for which it is decreasing.
- (iv) For which values of  $x$  between 0 and  $3\pi$  does  $\text{Si}(x)$  have stationary points?
- (v) Use the graph of  $f(t)$  to estimate  $\text{Si}(\pi)$ ,  $\text{Si}(2\pi)$  and  $\text{Si}(3\pi)$ .
- (vi) Use the graph of  $f(t)$  to determine values of  $x$  at which  $\text{Si}$  has points of inflection.
- (vii) Sketch the graph of  $\text{Si}$  for  $0 \leq x \leq 3\pi$ .



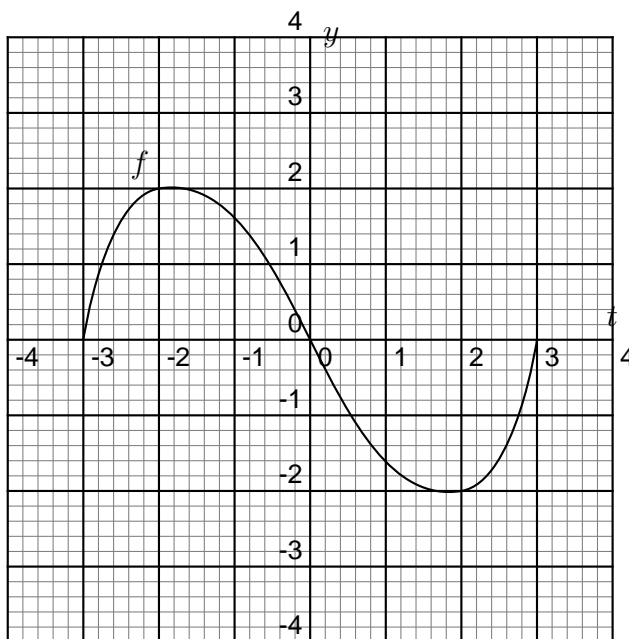
4. Establish the following reduction formula. [Hint: Write the integrand as  $u(x)v'(x)$  where  $u = \cos^{n-1} x$ .]

$$\int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx .$$

Use this formula to find  $\int \cos^2 x \, dx$  and  $\int \cos^4 x \, dx$ .

### More Questions

5. Let  $g(x) = \int_{-3}^x f(t) \, dt$  where  $f$  is the *odd* function whose graph is shown.



- (i) Evaluate  $g(-3)$  and  $g(3)$ .  
(ii) Estimate  $g(-2)$ ,  $g(-1)$  and  $g(0)$ .  
(iii) On what interval is  $g$  increasing?  
(iv) Where does  $g$  have a maximum value?  
(v) Sketch a rough graph of  $g$ .
6. Evaluate the following integrals by using integration by parts.
- (i)  $\int_0^1 (2x + 3)e^x dx$ .      (ii)  $\int_0^\pi \theta^2 \cos 3\theta d\theta$ .      (iii)  $\int_{-\pi/4}^{\pi/4} t \sin t \cos t dt$ .

Hint: First use an identity in (iii).

7. Let  $I_n = \int x^n e^x dx$ . Use integration by parts to establish the reduction formula

$$I_n = x^n e^x - nI_{n-1}.$$

Hence find  $\int x^3 e^x dx$ .

8. (i) Let  $I_n = \int x(\ln x)^n dx$ . Use integration by parts to establish the reduction formula

$$I_n = \frac{1}{2}x^2(\ln x)^n - \frac{n}{2}I_{n-1}.$$

- (ii) Starting with  $I_0 = \int x dx = \frac{1}{2}x^2 + C$ , use the reduction formula from part (i) to find  $I_2$ .

### Answers to Selected Questions

1. (i)  $-\ln |\cos x| + C$ .      (ii)  $(x^2 - 2x + 2)e^x + C$ .
2. (i)  $\frac{1}{4}$ .      (ii)  $\frac{\pi}{16}$ .      (iii)  $\frac{8}{3} \ln 2 - \frac{7}{9}$ .
3. (iii)  $\text{Si}(x)$  is increasing for  $0 < x < \pi$  and  $2\pi < x < 3\pi$ , and decreasing for  $\pi < x < 2\pi$ .  
(iv)  $\pi$ ,  $2\pi$  and  $3\pi$   
(v)  $\text{Si}(\pi) \approx 1.9$ ,  $\text{Si}(2\pi) \approx 1.4$ ,  $\text{Si}(3\pi) \approx 1.7$   
(vi) 0, 4.5, 7.7
4.  $I_2 = \frac{1}{2} \cos x \sin x + \frac{1}{2}x + C$
5. (i)  $g(-3) = g(3) = 0$  (ii)  $g(-2) \approx 1.4$ ,  $g(-1) \approx 3.3$ ,  $g(0) \approx 4.2$  (iii)  $(-3, 0)$  (iv)  $t = 0$
6. (i)  $3e - 1$ .      (ii)  $-\frac{2\pi}{9}$ .      (iii)  $\frac{1}{4}$ .
7.  $I_3 = x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + C$
8.  $I_2 = \frac{1}{2}x^2(\ln x)^2 - \frac{1}{2}x^2 \ln x + \frac{1}{4}x^2 + C$