

Assumed Knowledge Sketching curves of simple functions. Integrals of simple functions such as x^n (including $1/x$), $\sin x$, $\cos x$, e^x .

Objectives

- (4a) To understand and be able to use integration by parts to evaluate definite integrals.
- (4b) To understand that an indefinite integral is a function.
- (4c) To understand that differentiation and (indefinite) integration are inverse processes when applied to functions.
- (4d) To be able to sketch a function defined by an integral.
- (4e) To be able to derive a reduction formula for an integral.

Preparatory Questions

1. (i) Find the indefinite integral $\int \tan x \, dx$.

Hint: Use $\tan x = \frac{\sin x}{\cos x}$ and substitute $u = \cos x$.

- (ii) Use two integration by part to find the indefinite integral $\int x^2 e^x \, dx$.

Practice Questions

2. Evaluate the following integrals by using integration by parts.

(i) $\int_0^{1/2} x e^{2x} \, dx$. (ii) $\int_0^{\pi/4} \theta \sin 4\theta \, d\theta$. (iii) $\int_1^2 t^2 \ln t \, dt$.

Solution

- (i) Choose $u = x$ and $dv = e^{2x} dx$. Then $du = dx$ and $v = \frac{1}{2}e^{2x} dx$. So,

$$\int_0^{1/2} x e^{2x} \, dx = \left[x \frac{1}{2} e^{2x} \right]_0^{1/2} - \int_0^{1/2} \frac{1}{2} e^{2x} \, dx = \left[x \frac{1}{2} e^{2x} \right]_0^{1/2} - \left[\frac{1}{4} e^{2x} \right]_0^{1/2} = \frac{1}{4}.$$

- (ii) Choose $u = \theta$ and $dv = \sin 4\theta \, d\theta$. Then $du = d\theta$ and $v = -\frac{1}{4} \cos 4\theta$. So,

$$\begin{aligned} \int_0^{\pi/4} \theta \sin 4\theta \, d\theta &= \left[\theta \left(-\frac{1}{4} \cos 4\theta \right) \right]_0^{\pi/4} - \int_0^{\pi/4} (-\cos 4\theta) \, d\theta \\ &= \left[\theta \left(-\frac{1}{4} \cos 4\theta \right) \right]_0^{\pi/4} + \left[\frac{1}{16} \sin 4\theta \right]_0^{\pi/4} = \frac{\pi}{16}. \end{aligned}$$

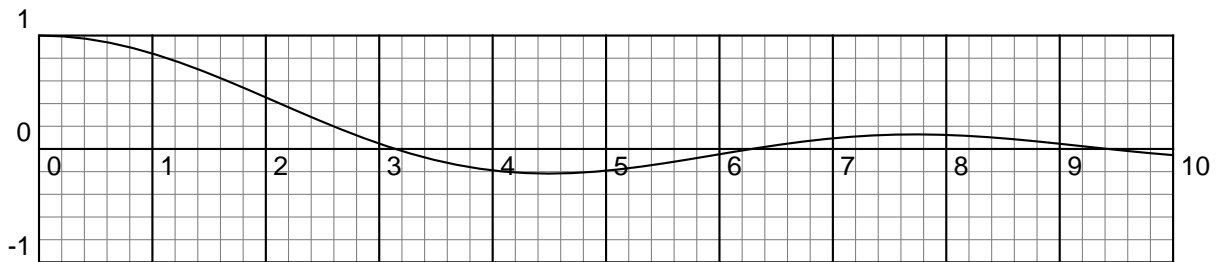
(iii)

$$\int_1^2 t^2 \ln t \, dt = \left[\frac{1}{3} t^3 \ln t \right]_1^2 - \int_1^2 \frac{1}{3} t^3 \frac{1}{t} dt = \left[\frac{1}{3} t^3 \ln t \right]_1^2 - \left[\frac{t^3}{9} \right]_1^2 = \frac{8}{3} \ln 2 - \frac{7}{9}.$$

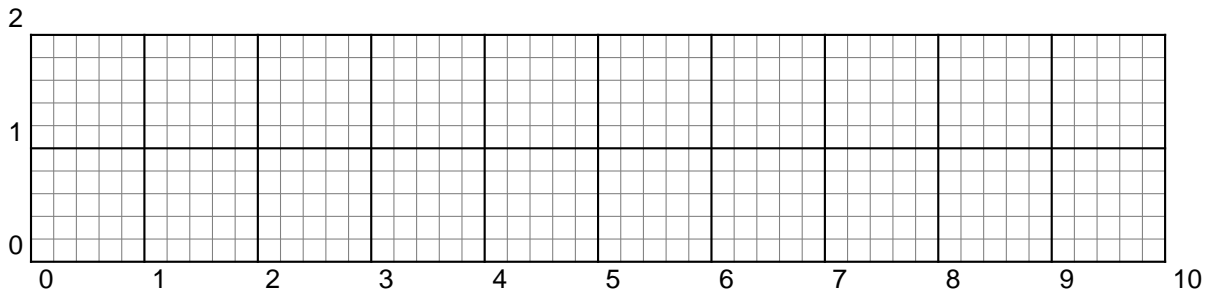
3. Define $\text{Si}(x)$ as $\text{Si}(x) = \int_0^x f(t) \, dt$, where $f(t) = \begin{cases} \frac{\sin t}{t} & t \neq 0 \\ 1 & t = 0. \end{cases}$

This function is called the *sine-integral*, and is useful in optics.

This is the graph of $f(t)$.



- (i) What is $\text{Si}'(x)$? (ii) What is the value of $\text{Si}(0)$?
(iii) For $0 \leq x \leq 3\pi$, use the graph of $f(t)$ to determine the values of x for which $\text{Si}(x)$ is increasing, and the values of x for which it is decreasing.
(iv) For which values of x between 0 and 3π does $\text{Si}(x)$ have stationary points?
(v) Use the graph of $f(t)$ to estimate $\text{Si}(\pi)$, $\text{Si}(2\pi)$ and $\text{Si}(3\pi)$.
(vi) Use the graph of $f(t)$ to determine values of x at which Si has points of inflection.
(vii) Sketch the graph of Si for $0 \leq x \leq 3\pi$.



Solution

- (i) By the Fundamental Theorem of Calculus Part I, $\text{Si}'(x) = \frac{\sin x}{x}$ for $x \neq 0$ and $\text{Si}'(x) = 1$ when $x = 0$.
(ii) $\text{Si}(0) = \int_0^0 \frac{\sin t}{t} \, dt = 0$.
(iii) From (a), we have $\text{Si}'(x) = (\sin x)/x$. Now $\text{Si}(x)$ is increasing when $\text{Si}'(x) > 0$, and decreasing when $\text{Si}'(x) < 0$, so we need to determine the values of x for which $(\sin x)/x$ is positive, and those for which it is negative. Looking at the graph provided, or noting that for $x > 0$ we have $(\sin x)/x > 0$ precisely when $\sin x > 0$ and $(\sin x)/x < 0$ precisely when $\sin x < 0$, we see that

$$\begin{aligned} (\sin x)/x &> 0, & \text{for } 0 < x < \pi \text{ and } 2\pi < x < 3\pi \\ (\sin x)/x &< 0, & \text{for } \pi < x < 2\pi. \end{aligned}$$

So $\text{Si}(x)$ is increasing for $0 < x < \pi$ and $2\pi < x < 3\pi$, and decreasing for $\pi < x < 2\pi$.

(iv) $\text{Si}(x)$ has stationary points when $\text{Si}'(x) = 0$. Looking at the graph again, or noting that $(\sin x)/x = 0$ precisely when $\sin x = 0$ (except at $x = 0$), we find the stationary points of $\text{Si}(x)$ to be at $x = \pi, 2\pi$, and 3π .

(v) Add up the squares under the graph of $(\sin x)/x$ to estimate $\text{Si}(x)$, remembering that the area of each square is $0.2 \times 0.2 = 0.04$ and that the area under the x axis is given a negative sign. This gives $\text{Si}(\pi) \doteq 1.9$, $\text{Si}(2\pi) \doteq 1.4$, $\text{Si}(3\pi) \doteq 1.7$. Note that these are approximate, and you may have obtained slightly different values.

(vi) Since $(\sin x)/x$ is the derivative of Si , the points of inflection of Si occur when $(\sin x)/x$ has stationary points. That is, when $x = 0, 4.5, 7.7$.

(vii) The sketch of the function is



4. Establish the following reduction formula. [Hint: Write the integrand as $u(x)v'(x)$ where $u = \cos^{n-1} x$.]

$$\int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx.$$

Use this formula to find $\int \cos^2 x \, dx$ and $\int \cos^4 x \, dx$.

Solution Putting $u = \cos^{n-1} x$ and $v' = \cos x$ so that $v = \sin x$, we can integrate by parts:

$$\begin{aligned} I_n &= \int \cos^n x \, dx = \int \cos^{n-1} x \frac{d}{dx}(\sin x) \, dx \\ &= \cos^{n-1} x \sin x - \int \sin x \frac{d}{dx}(\cos^{n-1} x) \, dx \\ &= \cos^{n-1} x \sin x - \int \sin x (n-1) \cos^{n-2} x (-\sin x) \, dx \\ &= \cos^{n-1} x \sin x + (n-1) \int \sin^2 x \cos^{n-2} x \, dx \\ &= \cos^{n-1} x \sin x + (n-1) \int (1 - \cos^2 x) \cos^{n-2} x \, dx \\ &= \cos^{n-1} x \sin x + (n-1) I_{n-2} - (n-1) I_n. \end{aligned}$$

Rearranging,

$$nI_n = \cos^{n-1} x \sin x + (n-1)I_{n-2} \quad \text{or} \quad I_n = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} I_{n-2}.$$

Now $I_0 = \int \cos^0 x \, dx = \int dx = x + C$, so

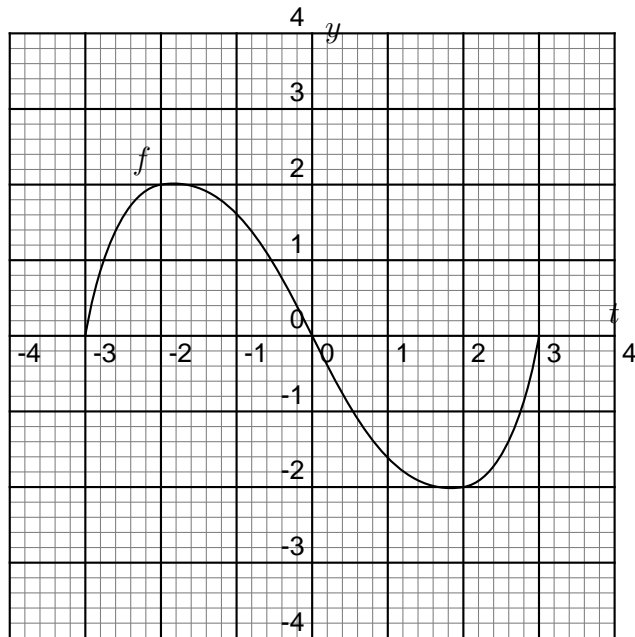
$$I_2 = \frac{1}{2} \cos x \sin x + \frac{1}{2} I_0 = \frac{1}{2} \cos x \sin x + \frac{1}{2} x + C_1$$

and

$$I_4 = \frac{1}{4} \cos^3 x \sin x + \frac{3}{4} I_2 = \frac{1}{4} \cos^3 x \sin x + \frac{3}{8} \cos x \sin x + \frac{3}{8} x + C_2.$$

More Questions

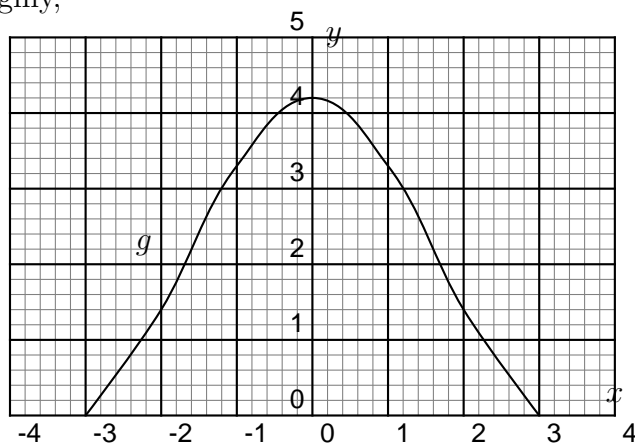
5. Let $g(x) = \int_{-3}^x f(t) dt$ where f is the *odd* function whose graph is shown.



- (i) Evaluate $g(-3)$ and $g(3)$.
- (ii) Estimate $g(-2)$, $g(-1)$ and $g(0)$.
- (iii) On what interval is g increasing?
- (iv) Where does g have a maximum value?
- (v) Sketch a rough graph of g .

Solution

- (i) $g(-3) = 0$, $g(3) = 0$.
- (ii) $g(-2) \approx 1.4$, $g(-1) \approx 3.3$, $g(0) \approx 4.2$.
- (iii) $(-3, 0)$.
- (iv) At $t = 0$.
- (v) The sketch is roughly,



6. Evaluate the following integrals by using integration by parts.

$$(i) \int_0^1 (2x + 3)e^x dx. \quad (ii) \int_0^\pi \theta^2 \cos 3\theta d\theta. \quad (iii) \int_{-\pi/4}^{\pi/4} t \sin t \cos t dt.$$

Hint: First use an identity in (iii).

Solution

(i)

$$\int_0^1 (2x + 3)e^x dx = [(2x + 3)e^x]_0^1 - \int_0^1 2e^x dx = [(2x + 3)e^x]_0^1 - [2e^x]_0^1 = 3e - 1.$$

(ii) First choose $u = \theta^2$ and $dv = \cos 3\theta d\theta$. So $du = 2\theta d\theta$ and $v = \frac{1}{3} \sin 3\theta$.

$$\begin{aligned} \int_0^\pi \theta^2 \cos 3\theta d\theta &= \left[\theta^2 \frac{1}{3} \sin 3\theta \right]_0^\pi - \int_0^\pi 2\theta \frac{1}{3} \sin 3\theta d\theta \\ &= \left[\frac{1}{3} \theta^2 \sin 3\theta \right]_0^\pi - \frac{2}{3} \int_0^\pi \theta \sin 3\theta d\theta. \end{aligned}$$

Now we need to integrate by parts again, this time choosing $u = \theta$ and $dv = \sin 3\theta d\theta$.

$$\begin{aligned}\int_0^\pi \theta \sin 3\theta d\theta &= \left[\theta \left(-\frac{1}{3} \cos 3\theta \right) \right]_0^\pi - \int_0^\pi \left(-\frac{1}{3} \cos 3\theta \right) d\theta \\ &= \left[-\frac{1}{3} \theta \cos 3\theta \right]_0^\pi + \left[\frac{1}{9} \sin 3\theta \right]_0^\pi.\end{aligned}$$

So

$$\int_0^\pi \theta^2 \cos 3\theta d\theta = \left[\frac{1}{3} \theta^2 \sin 3\theta + \frac{2}{9} \theta \cos 3\theta - \frac{2}{27} \sin 3\theta \right]_0^\pi = -\frac{2\pi}{9}.$$

(iii) First use the identity $\sin 2t = 2 \sin t \cos t$ and then integration by parts.

$$\begin{aligned}\int_{-\pi/4}^{\pi/4} t \sin t \cos t dt &= \frac{1}{2} \int_{-\pi/4}^{\pi/4} t \sin 2t dt \\ &= \frac{1}{2} \left(\left[t \left(-\frac{1}{2} \cos 2t \right) \right]_{-\pi/4}^{\pi/4} - \int_{-\pi/4}^{\pi/4} \left(-\frac{1}{2} \cos 2t \right) dt \right) \\ &= \left[-\frac{1}{4} t \cos 2t + \frac{1}{8} \sin 2t \right]_{-\pi/4}^{\pi/4} = \frac{1}{4}.\end{aligned}$$

7. Let $I_n = \int x^n e^x dx$. Use integration by parts to establish the reduction formula

$$I_n = x^n e^x - n I_{n-1}.$$

Hence find $\int x^3 e^x dx$.

Solution Choose $u = x^n$ and $dv/dx = e^x$, i.e. $v = e^x$. Then

$$I_n = \int x^n \frac{d}{dx} e^x dx = x^n e^x - \int n x^{n-1} e^x dx = x^n e^x - n I_{n-1}.$$

To find I_3 , first find $I_0 = \int x^0 e^x dx = \int e^x dx = e^x + C$. Then

$$I_1 = x e^x - I_0 = x e^x - e^x + C_1,$$

and

$$I_2 = x^2 e^x - 2 I_1 = x^2 e^x - 2 x e^x + 2 e^x + C_2,$$

and

$$I_3 = x^3 e^x - 3 I_2 = x^3 e^x - 3 x^2 e^x + 6 x e^x - 6 e^x + C_3.$$

8. (i) Let $I_n = \int x (\ln x)^n dx$. Use integration by parts to establish the reduction formula

$$I_n = \frac{1}{2} x^2 (\ln x)^n - \frac{n}{2} I_{n-1}.$$

(ii) Starting with $I_0 = \int x dx = \frac{1}{2} x^2 + C$, use the reduction formula from part (i) to find I_2 .

Solution

(i) Since we cannot easily integrate $(\ln x)^n$, we choose $u = (\ln x)^n$ and $v'(x) = x$, i.e. $v(x) = x^2/2$. Then

$$\begin{aligned} \int x(\ln x)^n dx &= \int (\ln x)^n \frac{d}{dx} \left(\frac{x^2}{2} \right) dx \\ &= \frac{1}{2}x^2(\ln x)^n - \int \frac{x^2}{2} \frac{d}{dx} (\ln x)^n dx \\ &= \frac{1}{2}x^2(\ln x)^n - \int \frac{x^2}{2} n(\ln x)^{n-1} \frac{1}{x} dx \\ &= \frac{1}{2}x^2(\ln x)^n - \frac{n}{2} \int x(\ln x)^{n-1} dx = \frac{1}{2}x^2(\ln x)^n - \frac{n}{2}I_{n-1}. \end{aligned}$$

(ii) Starting with $I_0 = \frac{1}{2}x^2 + C$,

$$I_1 = \frac{1}{2}x^2(\ln x) - \frac{1}{2}I_0 = \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C_1,$$

where C_1 is a constant. Applying the formula again,

$$I_2 = \frac{1}{2}x^2(\ln x)^2 - \frac{2}{2}I_1 = \frac{1}{2}x^2(\ln x)^2 - \frac{1}{2}x^2 \ln x + \frac{1}{4}x^2 + C_2.$$

Answers to Selected Questions

1. (i) $-\ln |\cos x| + C$. (ii) $(x^2 - 2x + 2)e^x + C$.
2. (i) $\frac{1}{4}$. (ii) $\frac{\pi}{16}$. (iii) $\frac{8}{3} \ln 2 - \frac{7}{9}$.
3. (iii) $\text{Si}(x)$ is increasing for $0 < x < \pi$ and $2\pi < x < 3\pi$, and decreasing for $\pi < x < 2\pi$.
 (iv) $\pi, 2\pi$ and 3π
 (v) $\text{Si}(\pi) \approx 1.9, \text{Si}(2\pi) \approx 1.4, \text{Si}(3\pi) \approx 1.7$
 (vi) $0, 4.5, 7.7$
4. $I_2 = \frac{1}{2} \cos x \sin x + \frac{1}{2}x + C$
5. (i) $g(-3) = g(3) = 0$ (ii) $g(-2) \approx 1.4, g(-1) \approx 3.3, g(0) \approx 4.2$ (iii) $(-3, 0)$ (iv) $t = 0$
6. (i) $3e - 1$. (ii) $-\frac{2\pi}{9}$. (iii) $\frac{1}{4}$.
7. $I_3 = x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C$
8. $I_2 = \frac{1}{2}x^2(\ln x)^2 - \frac{1}{2}x^2 \ln x + \frac{1}{4}x^2 + C$