

Assumed Knowledge Simple properties of the functions $\ln x$ and e^x , including their derivatives.

Objectives

- (5a) To know and be able to use the properties of the \ln function.
- (5b) To know and be able to use the properties of the \exp function.
- (5c) To know and be able to use the properties of the generalised exponential function a^x .
- (5d) To be able to perform a logarithmic differentiation.

Preparatory Questions

1. Simplify each of the following expressions:

(i) $e^{\ln 6}$ (ii) $\ln \sqrt{e}$ (iii) $e^{x+\ln x}$.

2. Find dy/dx for each of the following:

(i) $y = 2^x$ (ii) $y = \ln \left(\frac{1+x}{1-x} \right)$
(iii) $y = \log_2 x$ (iv) $y = \log_{10} x$

Practice Questions

3. Find dy/dx for each of the following:

(i) $y = 3 \log_2(x^2)$ (ii) $y = \log_{10} \sqrt{x}$
(iii) $y = x^x$ (iv) $y = (\sin x)^x$
(v) $y = \frac{x^2 \sqrt[3]{7x-4}}{(1+x^2)^4}$

4. Recall the hyperbolic sine and hyperbolic cosine functions, $\sinh x$ and $\cosh x$, are defined as $\sinh x = \frac{1}{2}(e^x - e^{-x})$ and $\cosh x = \frac{1}{2}(e^x + e^{-x})$. Using properties of the exponential function, show that

(i) $\frac{d}{dx} \sinh x = \cosh x$ (ii) $\frac{d}{dx} \cosh x = \sinh x$.
(iii) $\cosh A \cosh B + \sinh A \sinh B = \cosh(A+B)$
(iv) $2(\cosh A)^2 - 1 = \cosh(2A)$.

5. (i) Use l'Hôpital's rule to find $\lim_{x \rightarrow 0} \ln\{(1+x)^{1/x}\}$.
(ii) Hence find $\lim_{x \rightarrow 0} (1+x)^{1/x}$.
[Note: $\lim_{x \rightarrow 0} \ln\{(1+x)^{1/x}\} = \ln\left(\lim_{x \rightarrow 0} (1+x)^{1/x}\right)$. Can you say why?]
6. (i) For which positive real numbers x is it true that $\sqrt{x} > x/2$?
(ii) *Without* using your calculator, and assuming that $\pi \approx 3$, determine which of the following is bigger: $(\sqrt{\pi})^\pi$ or $\pi^{\sqrt{\pi}}$.
[Hint: The exponential function is always increasing, so if $a > b$, then $e^a > e^b$.]

More Questions

7. Consider the function $f(x) = \frac{\ln x}{x}$, which is defined for all $x > 0$. Show that this function is strictly increasing on the interval $(0, e)$, strictly decreasing on the interval (e, ∞) , and thus has a global maximum at $x = e$. Hence show that $f(x) \leq \frac{1}{e}$ for all $x > 0$. Use this result with $x = \pi$ to show that $\pi^e < e^\pi$.

Answers to Selected Questions

1. (i) $e^{\ln 6} = 6$.
(ii) $\ln \sqrt{e} = \ln(e^{1/2}) = \frac{1}{2} \ln e = \frac{1}{2}$.
(iii) $e^{x+\ln x} = e^x e^{\ln x} = x e^x$.
2. (i) $2^x \ln 2$
(ii) $\frac{2}{1-x^2}$
(iii) $\frac{1}{x \ln 2}$
(iv) $\frac{1}{x \ln 10}$
3. (i) $\frac{6}{x \ln 2}$
(ii) $\frac{1}{2x \ln 10}$
(iii) $x^x(1 + \ln x)$
(iv) $(\sin x)^x(x \cot x + \ln(\sin x))$
(v) $\frac{x^2 \sqrt[3]{7x-4}}{(1+x^2)^4} \left(\frac{2}{x} + \frac{7}{3(7x-4)} - \frac{8x}{1+x^2} \right)$
5. (i) 1
(ii) e
6. (i) $0 < x < 4$
(ii) $(\sqrt{\pi})^\pi < \pi^{\sqrt{\pi}}$