

**Assumed Knowledge** Simple properties of the functions  $\ln x$  and  $e^x$ , including their derivatives.

### Objectives

- (5a) To know and be able to use the properties of the  $\ln$  function.
- (5b) To know and be able to use the properties of the  $\exp$  function.
- (5c) To know and be able to use the properties of the generalised exponential function  $a^x$ .
- (5d) To be able to perform a logarithmic differentiation.

### Preparatory Questions

1. Simplify each of the following expressions:

(i)  $e^{\ln 6}$                       (ii)  $\ln \sqrt{e}$                       (iii)  $e^{x+\ln x}$ .

2. Find  $dy/dx$  for each of the following:

(i)  $y = 2^x$                       (ii)  $y = \ln \left( \frac{1+x}{1-x} \right)$   
(iii)  $y = \log_2 x$                       (iv)  $y = \log_{10} x$

### Practice Questions

3. Find  $dy/dx$  for each of the following:

(i)  $y = 3 \log_2(x^2)$                       (ii)  $y = \log_{10} \sqrt{x}$   
(iii)  $y = x^x$                       (iv)  $y = (\sin x)^x$   
(v)  $y = \frac{x^2 \sqrt[3]{7x-4}}{(1+x^2)^4}$

*Solution*

(i)  $3 \log_2(x^2) = 6 \log_2(x) = \frac{6 \ln x}{\ln 2}$  so  $\frac{dy}{dx} = \frac{6}{x \ln 2}$ .  
(ii)  $\log_{10} \sqrt{x} = \frac{1}{2} \log_{10}(x) = \frac{\ln x}{2 \ln 10}$  so  $\frac{dy}{dx} = \frac{1}{2x \ln 10}$ .  
(iii)  $\frac{dy}{dx} = \frac{d}{dx} (x^x) = \frac{d}{dx} (e^{x \ln x}) = e^{x \ln x} \left( x \times \frac{1}{x} + \ln x \right) = x^x(1 + \ln x)$ .

(iv) First write  $(\sin x)^x$  as  $e^{x \ln(\sin x)}$ , then

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} ((\sin x)^x) = \frac{d}{dx} (e^{x \ln(\sin x)}) \\ &= e^{x \ln(\sin x)} \left( x \times \frac{1}{\sin x} \times \cos x + \ln(\sin x) \right) \\ &= (\sin x)^x (x \cot x + \ln(\sin x)).\end{aligned}$$

(v)  $\ln y = 2 \ln x + \frac{1}{3} \ln(7x - 4) - 4 \ln(1 + x^2)$ .

Differentiating both sides with respect to  $x$ , we have

$$\frac{1}{y} \frac{dy}{dx} = \frac{2}{x} + \frac{1}{3} \times \frac{7}{7x - 4} - 4 \times \frac{2x}{1 + x^2}.$$

$$\text{Therefore, } \frac{dy}{dx} = \frac{x^2 \sqrt[3]{7x - 4}}{(1 + x^2)^4} \left( \frac{2}{x} + \frac{7}{3(7x - 4)} - \frac{8x}{1 + x^2} \right).$$

4. Recall the hyperbolic sine and hyperbolic cosine functions,  $\sinh x$  and  $\cosh x$ , are defined as  $\sinh x = \frac{1}{2}(e^x - e^{-x})$  and  $\cosh x = \frac{1}{2}(e^x + e^{-x})$ .

Using properties of the exponential function, show that

$$(i) \quad \frac{d}{dx} \sinh x = \cosh x \qquad (ii) \quad \frac{d}{dx} \cosh x = \sinh x.$$

$$(iii) \quad \cosh A \cosh B + \sinh A \sinh B = \cosh(A + B)$$

$$(iv) \quad 2(\cosh A)^2 - 1 = \cosh(2A).$$

*Solution*

$$(i) \quad \frac{d}{dx} \sinh x = \frac{1}{2} \frac{d}{dx} (e^x - e^{-x}) = \frac{1}{2} (e^x + e^{-x}) = \cosh x.$$

$$(ii) \quad \frac{d}{dx} \cosh x = \frac{1}{2} \frac{d}{dx} (e^x + e^{-x}) = \frac{1}{2} (e^x - e^{-x}) = \sinh x.$$

(iii)

$$\begin{aligned}\cosh A \cosh B + \sinh A \sinh B &= \frac{1}{4}(e^A + e^{-A})(e^B + e^{-B}) + \frac{1}{4}(e^A - e^{-A})(e^B - e^{-B}) \\ &= \frac{1}{4}(e^{A+B} + e^{A-B} + e^{-A+B} + e^{-(A+B)} + e^{A+B} - e^{A-B} - e^{-A+B} + e^{-(A+B)}) \\ &= \frac{1}{2}(e^{A+B} + e^{-(A+B)}) = \cosh(A + B).\end{aligned}$$

$$(iv) \quad 2 \left( \frac{e^A + e^{-A}}{2} \right)^2 - 1 = \frac{1}{2}(e^{2A} + 2e^A e^{-A} + e^{-2A}) - 1 = \frac{1}{2}(e^{2A} + e^{-2A}) = \cosh(2A).$$

5. (i) Use l'Hôpital's rule to find  $\lim_{x \rightarrow 0} \ln\{(1 + x)^{1/x}\}$ .

(ii) Hence find  $\lim_{x \rightarrow 0} (1 + x)^{1/x}$ .

[Note:  $\lim_{x \rightarrow 0} \ln\{(1 + x)^{1/x}\} = \ln \left( \lim_{x \rightarrow 0} (1 + x)^{1/x} \right)$ . Can you say why?]

*Solution*

(i)  $\lim_{x \rightarrow 0} \ln(1+x)^{1/x} = \lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = \lim_{x \rightarrow 0} \frac{1}{1+x} = 1$  where l'Hôpital's rule has been used in the second step.

(ii) Since  $\ln x$  is continuous,  $\lim_{x \rightarrow 0} \ln(1+x)^{1/x} = \ln \left( \lim_{x \rightarrow 0} (1+x)^{1/x} \right) = 1$ .

Therefore,  $\lim_{x \rightarrow 0} (1+x)^{1/x} = e$ .

6. (i) For which positive real numbers  $x$  is it true that  $\sqrt{x} > x/2$ ?

(ii) Without using your calculator, and assuming that  $\pi \approx 3$ , determine which of the following is bigger:  $(\sqrt{\pi})^\pi$  or  $\pi^{\sqrt{\pi}}$ .

[Hint: The exponential function is always increasing, so if  $a > b$ , then  $e^a > e^b$ .]

*Solution*

(i) If  $\sqrt{x} > x/2$  then  $4x > x^2$  or  $(4-x)x > 0$  which requires  $0 < x < 4$ .

(ii)  $(\sqrt{\pi})^\pi = \exp(\pi \ln \sqrt{\pi}) = \exp(\frac{\pi}{2} \ln \pi)$ , and  $\pi^{\sqrt{\pi}} = \exp(\sqrt{\pi} \ln \pi)$ .

Now,  $0 < \pi < 4$ , and so  $\frac{\pi}{2} < \sqrt{\pi}$  (by part (i)).

Therefore  $\frac{\pi}{2} \ln \pi < \sqrt{\pi} \ln \pi$  ( $\ln \pi > 0$ ), and  $\exp(\frac{\pi}{2} \ln \pi) < \exp(\sqrt{\pi} \ln \pi)$ .

That is,  $(\sqrt{\pi})^\pi < \pi^{\sqrt{\pi}}$ .

## More Questions

7. Consider the function  $f(x) = \frac{\ln x}{x}$ , which is defined for all  $x > 0$ . Show that this function is strictly increasing on the interval  $(0, e)$ , strictly decreasing on the interval  $(e, \infty)$ , and thus has a global maximum at  $x = e$ . Hence show that  $f(x) \leq \frac{1}{e}$  for all  $x > 0$ . Use this result with  $x = \pi$  to show that  $\pi^e < e^\pi$ .

*Solution* The derivative  $f' = \frac{1}{x} \left( \frac{1}{x} \right) - \frac{1}{x^2} \ln x = \frac{1 - \ln x}{x^2}$ . Since  $x^2 > 0$  for  $x > 0$ , the sign of  $f'(x)$  is determined by the sign of  $1 - \ln x$ . This is positive when  $1 - \ln x > 0$ , i.e.  $\ln x < 1$ , and negative when  $1 - \ln x < 0$ , i.e.  $\ln x > 1$ . Now  $\ln x = 1$  when  $e^{\ln x} = e^1$ , i.e. when  $x = e$ . Therefore, when  $0 < x < e$ ,  $f'(x) > 0$  so the function is increasing and, when  $x > e$ ,  $f'(x) < 0$  so the function is decreasing. Hence,  $f(x)$  has a maximum when  $x = e$ .

If  $f(x)$  has a maximum at  $x = e$ , then  $f(x) \leq \frac{\ln e}{e} = \frac{1}{e}$  since  $\ln e = 1$ .

Now  $\pi^e = e^{\ln \pi^e} = e^{e \ln \pi}$ . But if we put  $x = \pi$  in the previous result,  $\ln \pi / \pi < 1/e$  or  $e \ln \pi < \pi$  (an inequality holds because the maximum occurs at  $x = e$ , not  $x = \pi$ ). Since  $e^x$  is an increasing function of  $x$ ,  $e^{e \ln \pi} < e^\pi$ , i.e.  $\pi^e < e^\pi$ .

## Answers to Selected Questions

1. (i)  $e^{\ln 6} = 6$ .

(ii)  $\ln \sqrt{e} = \ln(e^{1/2}) = \frac{1}{2} \ln e = \frac{1}{2}$ .

(iii)  $e^{x+\ln x} = e^x e^{\ln x} = x e^x$ .

2. (i)  $2^x \ln 2$

(ii)  $\frac{2}{1-x^2}$

$$(iii) \frac{1}{x \ln 2}$$

$$(iv) \frac{1}{x \ln 10}$$

$$3. (i) \frac{6}{x \ln 2}$$

$$(ii) \frac{1}{2x \ln 10}$$

$$(iii) x^x(1 + \ln x)$$

$$(iv) (\sin x)^x(x \cot x + \ln(\sin x))$$

$$(v) \frac{x^2 \sqrt[3]{7x-4}}{(1+x^2)^4} \left( \frac{2}{x} + \frac{7}{3(7x-4)} - \frac{8x}{1+x^2} \right)$$

$$5. (i) 1$$

$$(ii) e$$

$$6. (i) 0 < x < 4$$

$$(ii) (\sqrt{\pi})^\pi < \pi^{\sqrt{\pi}}$$