The University of Sydney
Math1003 Integral Calculus and Modelling

Semester 2 **Exercises for Week 7** 2009

Assumed Knowledge Proportionality and inverse proportionality. Integration techniques. Taylor series expansion.

Objectives

(6a) Given a verbal description of a simple model, to be able to express it as a mathematical equation.

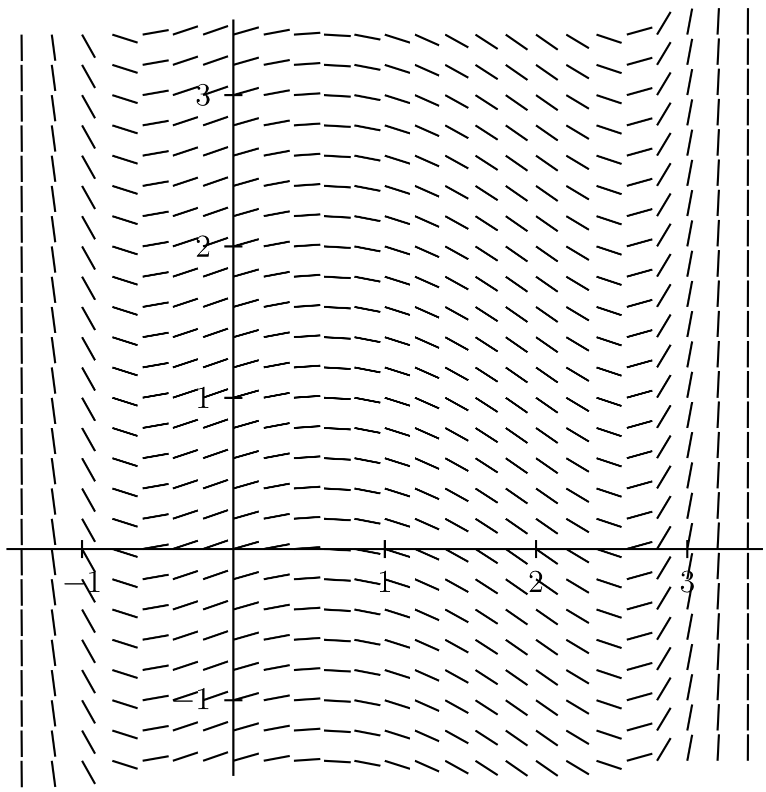
(6b) To be able to recognise an ordinary differential equation.

(6c) To be able to sketch the solution curves for a first-order differential equation from its direction field.

Preparatory Questions

1. (*i*) The differential equation $\dfrac{dy}{dx} = f(x)$ has a direction field given by the diagram below.
On the direction field draw the graphs of two solutions of $dy/dx = f(x)$, where one solution $y = g(x)$ passes through the point $(0, 1)$ and the other solution $y = h(x)$ satisfies the equation $h(1) = 0$.

(*ii*) Do the graphs of $y = g(x)$ and $y = h(x)$ intersect? If not, why not?



Practice Questions

2. Heat tends to flow from hot bodies to cold bodies. Newton observed that the rate at which temperature rises or falls within a body is proportional to the temperature difference between the body and its surroundings.

Consider a body warming or cooling in a room kept at constant temperature. Construct a mathematical model of this situation as follows. First define appropriate independent and dependent variables. Then determine a differential equation in these variables and any other constants you may need to introduce.

3. (*i*) Find the general solution by antidifferentiation and sketch the solution curves of:

(a) $\dfrac{dy}{dx} = \cos 2x$ (b) $\dfrac{dy}{dx} = \cosh x$

(*ii*) Find the particular solution of

$$\frac{dy}{dx} = \frac{1}{1+x^2}, \quad y(1) = \pi/4\,.$$

4. (Suitable for group work and discussion.) Newton's law of gravitation states that the acceleration of an object at a distance r from the centre of an object of mass M is given by

$$\frac{d^2r}{dt^2} = -\frac{GM}{r^2}\,,$$

where G is the universal gravitational constant.

(*i*) Use the identity

$$\frac{d^2r}{dt^2} = \frac{d}{dr}\left(\frac{1}{2}v^2\right),$$

where $v = dr/dt$, to show by integration that

$$v^2 - u^2 = \frac{2GM}{r} - \frac{2GM}{R}\,,$$

when $v = u$ at $r = R$.

(*ii*) Now write $r = R + s$ where s is the height of the object above the surface of the Earth, radius R and mass M. Use the Taylor series expansion

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots$$

with $x = s/R$ to show that, close to the surface of the Earth,

$$v^2 \simeq u^2 - 2gs\,,$$

for some constant g.

Find the expression for g in terms of G, M and R.

More Questions

5. (*i*) Find the general solution of the following differential equations, and in each case give also the particular solution satisfying the initial condition $y(0) = 2$.

(a) $\dfrac{dy}{dx} = 20xe^{5x^2}$ (b) $\dfrac{dy}{dx} = 6x^2 + \cos x$

(c) $\dfrac{dy}{dx} = x\cos x$ (d) $\dfrac{dy}{dx} = x^2e^x$

(*ii*) (a) Show that $y = A(e^x - x)$, where A is an arbitrary constant, satisfies the differential equation

$$\frac{dy}{dx} = \frac{y(e^x - 1)}{e^x - x}.$$

(b) Since the general solution of a first order-differential equation involves only one arbitrary constant, we see that the solution given in part (*ii*) (a) is the general solution of the differential equation. Find the particular solution satisfying the initial condition $y(0) = -3$.

(c) Sketch the solution curves given in part (*ii*) (a) in the special cases $A = 1$, $A = 0$ and $A = -1$.

6. *Please note that this question is a little harder than the previous ones.*
A ball has the property that each time it strikes a hard, level surface with velocity v it rebounds with velocity $-kv$, where $0 < k < 1$. Suppose that the ball is dropped from an initial height H above the surface.

(*i*) Recall that if s is the distance travelled upwards in time t then $s = ut - gt^2/2$, where u is the initial velocity, and g is the acceleration due to gravity. Use this formula to find the time t taken for the ball to hit the surface.

(*ii*) The formula $v = u - gt$ gives the velocity after time t of a body with initial velocity u when displacement is measured upwards. Use this to find the velocity with which it hits the surface.

(*iii*) Find the velocity with which it rebounds.

(*iv*) Use the formula of part (*i*) again to find the time taken for the ball to fall back to the surface after the first bounce.

(*v*) What is the speed of the ball just before the second bounce and what is the rebound speed after the second bounce? Find the time taken to fall back to the surface after the second bounce.

(*vi*) Write down the series for the time taken from the moment the ball is dropped until the fifth bounce.

(*vii*) How long is it before the ball comes to rest?

Answers to Selected Questions

1. (*i*) The sketch will be reproduced on the solution sheets.

(*ii*) No, the two curves do not intersect. They are shifted vertically by a constant amount.

3. (*i*) (a) $\frac{1}{2}\sin 2x + C$ (b) $\sinh x + C$

(*ii*) $y = \tan^{-1} x$

5. (*i*) (a) $y = 2e^{5x^2}$ (b) $y = 2x^3 + \sin x + 2$ (c) $y = x \sin x + \cos x + 1$
(d) $y = x^2 e^x - 2xe^x + 2e^x$

(*ii*) (b) $y = -3(e^x - x)$

6. (*i*) Time $\sqrt{2H/g}$ (*ii*) Velocity $\sqrt{2gH}$, downwards

(*iii*) Velocity $k\sqrt{2gH}$, upwards (*iv*) Time $2k\sqrt{2H/g}$

(*v*) Impact speed $2k^2\sqrt{2gH}$ (*vi*) Time $\sqrt{\frac{2H}{g}} + 2k\sqrt{\frac{2H}{g}}\left(1 + k + k^2 + k^3\right)$

(*vii*) After time $\sqrt{\frac{2H}{g}}\left(\frac{1+k}{1-k}\right)$