The University of Sydney
Math1003 Integral Calculus and Modelling

Semester 2 **Exercises and Solutions for Week 7** 2009

Assumed Knowledge Proportionality and inverse proportionality. Integration techniques. Taylor series expansion.

Objectives

(6a) Given a verbal description of a simple model, to be able to express it as a mathematical equation.

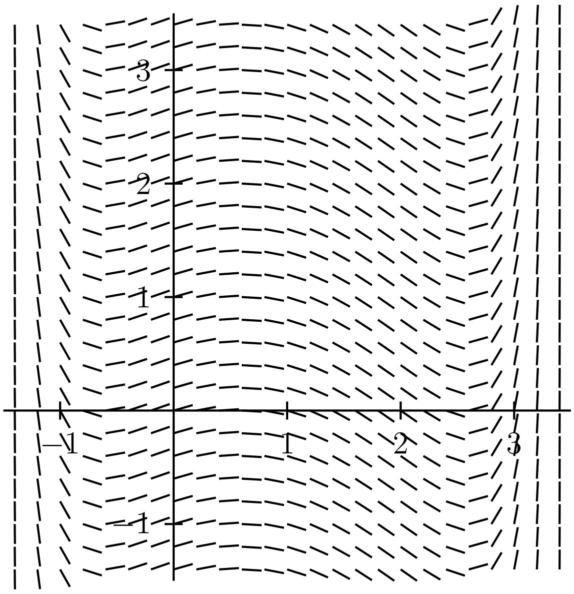
(6b) To be able to recognise an ordinary differential equation.

(6c) To be able to sketch the solution curves for a first-order differential equation from its direction field.

Preparatory Questions

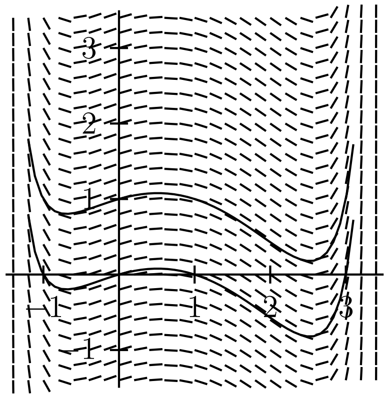
1. (*i*) The differential equation $\dfrac{dy}{dx} = f(x)$ has a direction field given by the diagram below.
On the direction field draw the graphs of two solutions of $dy/dx = f(x)$, where one solution $y = g(x)$ passes through the point $(0, 1)$ and the other solution $y = h(x)$ satisfies the equation $h(1) = 0$.

(*ii*) Do the graphs of $y = g(x)$ and $y = h(x)$ intersect? If not, why not?



Solution

(*i*)



(ii) No, because the second curve is displaced vertically by the same amount for all x.

Practice Questions

2. Heat tends to flow from hot bodies to cold bodies. Newton observed that the rate at which temperature rises or falls within a body is proportional to the temperature difference between the body and its surroundings.

Consider a body warming or cooling in a room kept at constant temperature. Construct a mathematical model of this situation as follows. First define appropriate independent and dependent variables. Then determine a differential equation in these variables and any other constants you may need to introduce.

Solution We are looking at temperature as a function of time. Let t be an independent variable measuring time and $T = T(t)$ be a dependent variable measuring the temperature of the body as a function of time t. Let T_s be the constant temperature of the room. Then Newton's observation is that dT/dt is proportional to $T - T_s$ and has the opposite sign to $T - T_s$. In mathematical terms then Newton's observation is

$$dT/dt = -k(T - Ts)$$

for some positive constant k.

3. (i) Find the general solution by antidifferentiation and sketch the solution curves of:

(a) $\dfrac{dy}{dx} = \cos 2x$ (b) $\dfrac{dy}{dx} = \cosh x$

(ii) Find the particular solution of

$$\frac{dy}{dx} = \frac{1}{1+x^2}, \quad y(1) = \pi/4\,.$$

Solution

(i) (a) $y = \int \cos 2x\,dx = \frac{1}{2}\sin 2x + C$. The set of solution curves is a set of sine curves, each with amplitude $\frac{1}{2}$ and period π, displaced vertically one above the other.

(b) $y = \int \cosh x\,dx = \sinh x + C$. The set of solution curves is obtained by displacing $\sinh x$ vertically.

(ii)

$$y = \int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C\,.$$

This is the general solution. Putting $y = \pi/4$ when $x = 1$ gives $\pi/4 = \tan^{-1} 1 + C$, whence $C = 0$. Hence the required particular solution is $y = \tan^{-1} x$.

4. (Suitable for group work and discussion.) Newton's law of gravitation states that the acceleration of an object at a distance r from the centre of an object of mass M is given by

$$\frac{d^2r}{dt^2} = -\frac{GM}{r^2},$$

where G is the universal gravitational constant.

(i) Use the identity

$$\frac{d^2r}{dt^2} = \frac{d}{dr}\left(\frac{1}{2}v^2\right),$$

where $v = dr/dt$, to show by integration that

$$v^2 - u^2 = \frac{2GM}{r} - \frac{2GM}{R},$$

when $v = u$ at $r = R$.

(ii) Now write $r = R + s$ where s is the height of the object above the surface of the Earth, radius R and mass M. Use the Taylor series expansion

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots$$

with $x = s/R$ to show that, close to the surface of the Earth,

$$v^2 \simeq u^2 - 2gs,$$

for some constant g.
Find the expression for g in terms of G, M and R.

Solution

(i) Substituting,

$$\frac{d}{dr}\left(\frac{1}{2}v^2\right) = -\frac{GM}{r^2},$$

and performing the antidifferentiation,

$$\frac{1}{2}v^2 = -\int \frac{GM}{r^2}\,dr = \frac{GM}{r} + C.$$

We must choose C so that $v = u$ when $r = R$; so substituting, we want

$$\frac{1}{2}u^2 = \frac{GM}{R} + C.$$

Replacing C by this expression gives (after rearrangement)

$$v^2 - u^2 = \frac{2GM}{r} - \frac{2GM}{R}.$$

(ii) Writing $r = R + s$,

$$v^2 - u^2 = \frac{2GM}{(R+s)} - \frac{2GM}{R} = \frac{2GM}{R}\left(1 + \frac{s}{R}\right)^{-1} - \frac{2GM}{R}.$$

Using $(1+s/R)^{-1} = 1 - s/R + (s/R)^2 - (s/R)^3 + \ldots$, we obtain

$$v^2 = u^2 + \frac{2GM}{R}\left(1 - \frac{s}{R} + \frac{s^2}{R^2}\ldots\right) - \frac{2GM}{R} = u^2 - \frac{2GMs}{R^2}\ldots$$

Retaining only the leading term,

$$v^2 = u^2 - 2\left(\frac{GM}{R^2}\right)s\,,$$

so $g = GM/R^2$.

More Questions

5. (*i*) Find the general solution of the following differential equations, and in each case give also the particular solution satisfying the initial condition $y(0) = 2$.

(a) $\dfrac{dy}{dx} = 20xe^{5x^2}$ (b) $\dfrac{dy}{dx} = 6x^2 + \cos x$

(c) $\dfrac{dy}{dx} = x\cos x$ (d) $\dfrac{dy}{dx} = x^2e^x$

(*ii*) (a) Show that $y = A(e^x - x)$, where A is an arbitrary constant, satisfies the differential equation

$$\frac{dy}{dx} = \frac{y(e^x - 1)}{e^x - x}.$$

(b) Since the general solution of a first order-differential equation involves only one arbitrary constant, we see that the solution given in part (*ii*) (a) is the general solution of the differential equation. Find the particular solution satisfying the initial condition $y(0) = -3$.

(c) Sketch the solution curves given in part (*ii*) (a) in the special cases $A = 1$, $A = 0$ and $A = -1$.

Solution

(*i*) (a) General solution is $y = 2e^{5x^2} + C$. The condition $y(0) = 2$ then requires $2 = 2e^{5\times 0^2} + C = 2 + C$, so $C = 0$, and the particular solution is $y = 2e^{5x^2}$.

(b) General solution is $y = 2x^3 + \sin x + C$. The condition $y(0) = 2$ then requires $2 = C$, so the particular solution is $y = 2x^3 + \sin x + 2$.

(c) Use integration by parts to get the general solution

$$y = x\sin x - \int \sin x\, dx = x\sin x + \cos x + C.$$

The initial condition $y(0) = 2$ then requires $2 = 1 + C$, so $C = 1$, and the particular solution is $y = x\sin x + \cos x + 1$.

(d) Using integration by parts (twice) we get the general solution

$$y = x^2e^x - \int 2xe^x\, dx = x^2e^x - \left(2xe^x - \int 2e^x\, dx\right)$$
$$= x^2e^x - 2xe^x + 2e^x + C.$$

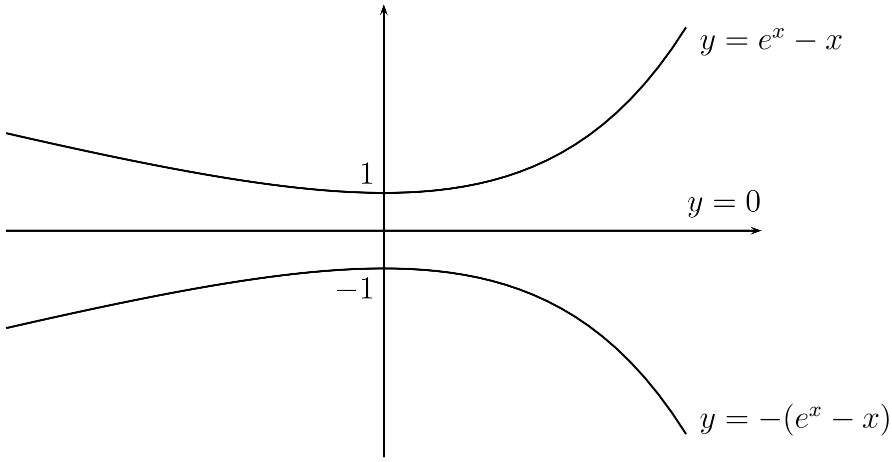
The initial condition $y(0) = 2$ the requires $2 = 2 + C$, so $C = 0$, and the particular solution is $y = x^2e^x - 2xe^x + 2e^x$.

(*ii*) (a) Differentiate $y = A(e^x - x)$ to get LHS $= \dfrac{dy}{dx} = A(e^x - 1)$. When we substitute y into the RHS we find $\dfrac{y(e^x - 1)}{e^x - x} = A(e^x - 1) =$ LHS, and so the given function satisfies the equation.

(b) We require that $y(0) = -3$, so we must have $-3 = A(e^0 - 0) = A$, and the particular solution is $y = -3(e^x - x)$.

(c) Consider first the case $A = 1$, $y = (e^x - x)$. Note that $y(0) = 1$. Also, $\dfrac{dy}{dx} = (e^x - 1)$ and so $\dfrac{dy}{dx} = 0$ when $x = 0$. Then since $\dfrac{dy}{dx} < 0$ for $x < 0$ and $\dfrac{dy}{dx} > 0$ for $x > 0$, we see that $x = 0$ is a global minimum. Also note that

$\dfrac{d^2y}{dx^2} = e^x > 0$ for all x.
Now consider the case $A = -1$, $y = -(e^x - x)$. Note that $y(0) = -1$. Also, $\dfrac{dy}{dx} = -(e^x - 1)$ and so $\dfrac{dy}{dx} = 0$ when $x = 0$. Then since $\dfrac{dy}{dx} > 0$ for $x < 0$ and $\dfrac{dy}{dx} < 0$ for $x > 0$, we see that $x = 0$ is a global maximum. Also note that $\dfrac{d^2y}{dx^2} = -e^x < 0$ for all x.
Finally, in the case $A = 0$ we obtain the particular solution $y = 0$.



6. *Please note that this question is a little harder than the previous ones.*
A ball has the property that each time it strikes a hard, level surface with velocity v it rebounds with velocity $-kv$, where $0 < k < 1$. Suppose that the ball is dropped from an initial height H above the surface.

(i) Recall that if s is the distance travelled upwards in time t then $s = ut - gt^2/2$, where u is the initial velocity, and g is the acceleration due to gravity. Use this formula to find the time t taken for the ball to hit the surface.

(ii) The formula $v = u - gt$ gives the velocity after time t of a body with initial velocity u when displacement is measured upwards. Use this to find the velocity with which it hits the surface.

(iii) Find the velocity with which it rebounds.

(iv) Use the formula of part (i) again to find the time taken for the ball to fall back to the surface after the first bounce.

(v) What is the speed of the ball just before the second bounce and what is the rebound speed after the second bounce? Find the time taken to fall back to the surface after the second bounce.

(vi) Write down the series for the time taken from the moment the ball is dropped until the fifth bounce.

(vii) How long is it before the ball comes to rest?

Solution

(i) Measuring s upwards from point of release $u = 0$, so time to $s = -H$ is given by $H = gt^2/2$, i.e. $t = \sqrt{2H/g}$ seconds.

(ii) After time $t = \sqrt{2H/g}$ seconds, $v = -g\sqrt{2H/g} = -\sqrt{2gH}$. The ball is travelling downwards with speed $\sqrt{2gH}$.

(*iii*) Rebound velocity is $k\sqrt{2gH}$ upwards.

(*iv*) Measuring s upwards from the point of bounce, $u = k\sqrt{2gH}$. The ball returns to the surface when $s = 0$, after a time $t > 0$ such that $0 = k(\sqrt{2gH})t - gt^2/2$. This has positive solution $t = 2k\sqrt{2gH}/g = 2k\sqrt{2H/g}$. The solution $t = 0$ is the time when the ball leaves the the surface.

(*v*) Now use formula of (*ii*) with $u = k\sqrt{2gH}$ and $t2k\sqrt{2H/g}$ to determine the velocity v of the ball as it returns to the surface. The formula gives $v = k\sqrt{2gH} - g2k\sqrt{2H/g} = -k\sqrt{2gH}$. We find that the ball is travelling downwards at the original rebound speed just before the second bounce. The rebound *speed* after the second bounce is $k \times k\sqrt{2gH} = k^2\sqrt{2gH}$. The calculation of the bounce time is now exactly as before. The time taken to fall back is thus $2k^2\sqrt{2H/g}$. Also as before and the speed on impact will be the same as it was on take-off, $k^2\sqrt{2gH}$.

(*vi*) The total time for initial drop, first, second, third and fourth bounces is

$$\sqrt{\frac{2H}{g}} + 2k\sqrt{\frac{2H}{g}} + 2k^2\sqrt{\frac{2H}{g}} + 2k^3\sqrt{\frac{2H}{g}} + 2k^4\sqrt{\frac{2H}{g}}$$
$$= \sqrt{\frac{2H}{g}} + 2k\sqrt{\frac{2H}{g}}\left(1 + k + k^2 + k^3\right) .$$

(*vii*) The total time taken for the ball to come to rest is

$$\sqrt{\frac{2H}{g}} + 2k\sqrt{\frac{2H}{g}}\sum_{n=0}^{\infty} k^n = \sqrt{\frac{2H}{g}} + 2k\sqrt{\frac{2H}{g}}\left(\frac{1}{1-k}\right) = \sqrt{\frac{2H}{g}}\left(\frac{1+k}{1-k}\right) .$$

So the ball comes to rest in a finite time.

Answers to Selected Questions

1. (*i*) The sketch will be reproduced on the solution sheets.

(*ii*) No, the two curves do not intersect. They are shifted vertically by a constant amount.

3. (*i*) (a) $\frac{1}{2}\sin 2x + C$ (b) $\sinh x + C$

(*ii*) $y = \tan^{-1} x$

5. (*i*) (a) $y = 2e^{5x^2}$ (b) $y = 2x^3 + \sin x + 2$ (c) $y = x\sin x + \cos x + 1$
(d) $y = x^2e^x - 2xe^x + 2e^x$

(*ii*) (b) $y = -3(e^x - x)$

6. (*i*) Time $\sqrt{2H/g}$ (*ii*) Velocity $\sqrt{2gH}$, downwards

(*iii*) Velocity $k\sqrt{2gH}$, upwards (*iv*) Time $2k\sqrt{2H/g}$

(*v*) Impact speed $2k^2\sqrt{2gH}$ (*vi*) Time $\sqrt{\frac{2H}{g}} + 2k\sqrt{\frac{2H}{g}}\left(1 + k + k^2 + k^3\right)$

(*vii*) After time $\sqrt{\frac{2H}{g}}\left(\frac{1+k}{1-k}\right)$