THE UNIVERSITY OF SYDNEY
MATH1003 INTEGRAL CALCULUS AND MODELLING

Semester 2 **Exercises for Week 8** 2009

Assumed Knowledge Factorisation of expressions. Simple techniques of integration.

Objectives

(7a) To be able to recognise a differential equation as a separable equation.

(7b) To be able to solve a separable equation by separation of variables.

(7c) To be able to perform integrations using trigonometric substitutions.

Preparatory Questions

1. Which of the following differential equations are separable? Write those that are in separated form.

$(i)\ \dfrac{dy}{dx} = \dfrac{x^2 y}{(x^2+1)^{1/2}}$ $\qquad (ii)\ \dfrac{dy}{dx} = \dfrac{a^2 e^y}{(a^2-x^2)^{3/2}} - \dfrac{e^y}{(a^2-x^2)^{1/2}}$

$(iii)\ \dfrac{dy}{dx} = \dfrac{x+\cos y}{x^3\sqrt{x^2-16}}.$

Practice Questions

2. Find the general solutions of

$(i)\ \dfrac{dy}{dx} = 1+y^2 \qquad (ii)\ \dfrac{dy}{dx} = y\cos x \qquad (iii)\ (1+x)\dfrac{dy}{dx} + y^2 = 0$

3. Evaluate the following integrals by making the given substitution:

$(i)\ \displaystyle\int \frac{x^2}{(a^2-x^2)^{3/2}}dx, \qquad x = a\sin u.$

$(ii)\ \displaystyle\int \frac{x^2}{(x^2+1)^{1/2}}dx, \qquad x = \sinh t.$

4. (Suitable for group work and discussion.) An animal population has a net growth rate per unit population which varies with the seasons, being positive in summer and negative in winter. Let $x(t)$ be the size of the population at time t, which is measured in years. The following differential equation is suggested as a model for this situation:

$$\frac{dx}{dt} = (k\cos 2\pi t)x \quad (k \text{ a positive constant}).$$

(i) What is the period of $\cos 2\pi t$?

(ii) What time of year do you think $t = 0$ represents ?

(*iii*) Can you explain why x has been multiplied by $(k \cos 2\pi t)$ in this model?

(*iv*) Solve the equation to find $x(t)$, given that $x = x_0$ at $t = 0$.

(*v*) Does $x(t)$ have a limiting value as $t \to \infty$?

(*vi*) What are the maximum and minimum values of x and when do they occur?

More Questions

5. (*i*) Find the general solution of

$$\frac{dy}{dx} = \frac{\operatorname{cosec} y}{\sqrt{x^2 + 4}}.$$

(*ii*) Find the particular solution of

$$\frac{dx}{dt} = \frac{x^2}{\cos^2 t}$$

if $x = 1$ when $t = 0$.

6. Find the general solutions of all the separable equations in Question 1.

7. (*i*) Given $y = A\sqrt{x^2 + 1}$, where A is an arbitrary constant, show by substitution that it satisfies the differential equation $\dfrac{dy}{dx} = \dfrac{xy}{x^2 + 1}$.

(*ii*) Since the general solution of a first-order differential equation depends on one arbitrary constant, we see that the solution given in part (*i*) is the general solution of $\dfrac{dy}{dx} = \dfrac{xy}{x^2 + 1}$. Now find the particular solution satisfying the initial condition $y(0) = 1$.

(*iii*) Sketch the family of solution curves given in part (*i*), indicating the behaviour of solutions for $A > 0$, $A = 0$ and $A < 0$. Indicate also on your sketch the particular solution found in part (*ii*).

Answers to Selected Questions

1. (*i*) Separable: $\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{x^2}{(x^2 + 1)^{1/2}}$. (*ii*) Separable: $e^{-y}\dfrac{dy}{dx} = \dfrac{x^2}{(a^2 - x^2)^{3/2}}$.

(*iii*) Not separable.

2. (*i*) $y = \tan(x - C)$ (*ii*) $y = Ae^{\sin x}$ (*iii*) $y = \dfrac{1}{\ln|1 + x| + C}$

3. (*i*) $\dfrac{x}{\sqrt{a^2 - x^2}} - \sin^{-1}\dfrac{x}{a} + C$ (*ii*) $\dfrac{1}{2}x\sqrt{1 + x^2} - \dfrac{1}{2}\sinh^{-1} x + C$

5. (*i*) $-\cos y = \sinh^{-1}\dfrac{x}{2} + C$ (*ii*) $x = \dfrac{1}{1 - \tan t}$

6. (*i*) $\ln|y| = \dfrac{1}{2}x\sqrt{1 + x^2} - \dfrac{1}{2}\sinh^{-1} x + C$ (*ii*) $-e^{-y} = \dfrac{x}{\sqrt{a^2 - x^2}} - \sin^{-1}\dfrac{x}{a} + C$