THE UNIVERSITY OF SYDNEY
MATH1003 INTEGRAL CALCULUS AND MODELLING

Semester 2 **Exercises and Solutions for Week 8** 2009

Assumed Knowledge Factorisation of expressions. Simple techniques of integration.

Objectives

(7a) To be able to recognise a differential equation as a separable equation.

(7b) To be able to solve a separable equation by separation of variables.

(7c) To be able to perform integrations using trigonometric substitutions.

Preparatory Questions

1. Which of the following differential equations are separable? Write those that are in separated form.

(i) $\dfrac{dy}{dx} = \dfrac{x^2 y}{(x^2+1)^{1/2}}$ *(ii)* $\dfrac{dy}{dx} = \dfrac{a^2 e^y}{(a^2-x^2)^{3/2}} - \dfrac{e^y}{(a^2-x^2)^{1/2}}$

(iii) $\dfrac{dy}{dx} = \dfrac{x+\cos y}{x^3\sqrt{x^2-16}}$.

Practice Questions

2. Find the general solutions of

(i) $\dfrac{dy}{dx} = 1 + y^2$ *(ii)* $\dfrac{dy}{dx} = y\cos x$ *(iii)* $(1+x)\dfrac{dy}{dx} + y^2 = 0$

Solution

(i) $x = \displaystyle\int \frac{1}{1+y^2}\,dy = \tan^{-1} y + C$, so $y = \tan(x - C)$.

(ii) $\displaystyle\int \frac{1}{y}\,dy = \int \cos x\,dx$ so $\ln|y| = \sin x + C$.
Therefore, $|y| = \exp(\sin x + C)$ or $y = Ae^{\sin x}$ (with $A = \pm e^C$). Note that we can allow $A = 0$ because $y = 0$ satisfies the original equation.

(iii) $\displaystyle\int -\frac{1}{y^2}\,dy = \int \frac{1}{1+x}\,dx$ so $\dfrac{1}{y} = \ln|1+x| + C$, and $y = \dfrac{1}{\ln|1+x|+C}$. In this case there is also a solution $y = 0$ which satisfies the original equation and is included in the general solution only in the limit $C \to \infty$.

3. Evaluate the following integrals by making the given substitution:

(i) $\displaystyle\int \frac{x^2}{(a^2-x^2)^{3/2}}dx,\qquad x = a\sin u.$

(*ii*) $\displaystyle\int \frac{x^2}{(x^2+1)^{1/2}}dx, \qquad x = \sinh t.$

Solution

(*i*) If $x = a\sin u$, then $dx = a\cos u\,du$ and so

$$\begin{aligned}\int \frac{x^2}{(a^2-x^2)^{3/2}}dx &= \int \frac{a^2\sin^2 u\, a\,\cos u\,du}{(a^2-a^2\sin^2 u)^{3/2}} = \int \tan^2 u\,du\\ &= \int (\sec^2 u - 1)\,du = \tan u - u + C\\ &= \frac{x}{\sqrt{a^2-x^2}} - \sin^{-1}\frac{x}{a} + C\,.\end{aligned}$$

(*ii*) If $x = \sinh t$, then $dx = \cosh t\,dt$ and so

$$\begin{aligned}\int \frac{x^2}{(x^2+1)^{1/2}}dx &= \int \frac{\sinh^2 t\cosh t}{(\cosh^2 t)^{1/2}}dx = \int \sinh^2 t\,dt\\ &= \int \frac{1}{2}(\cosh 2t - 1)\,dt = \frac{1}{2}\left(\frac{1}{2}\sinh 2t - t\right) + C\\ &= \frac{1}{2}x\sqrt{1+x^2} - \frac{1}{2}\sinh^{-1} x + C\,.\end{aligned}$$

4. (Suitable for group work and discussion.) An animal population has a net growth rate per unit population which varies with the seasons, being positive in summer and negative in winter. Let $x(t)$ be the size of the population at time t, which is measured in years. The following differential equation is suggested as a model for this situation:

$$\frac{dx}{dt} = (k\cos 2\pi t)x \quad (k \text{ a positive constant}).$$

(*i*) What is the period of $\cos 2\pi t$?

(*ii*) What time of year do you think $t = 0$ represents ?

(*iii*) Can you explain why x has been multiplied by $(k\cos 2\pi t)$ in this model?

(*iv*) Solve the equation to find $x(t)$, given that $x = x_0$ at $t = 0$.

(*v*) Does $x(t)$ have a limiting value as $t \to \infty$?

(*vi*) What are the maximum and minimum values of x and when do they occur?

Solution

(*i*) The period is one (year).

(*ii*) When $t = 0$, $\cos 2\pi t$ takes its maximum value of 1. Since k is positive, $k\cos 2\pi t$ is a maximum at $t = 0$, and so the population has its greatest rate of increase. Usually this is late spring or early summer.

(*iii*) The factor k is a constant of proportionality, and the situation described by this model is that the rate of change of population size is proportional to $\cos 2\pi t \times x$. The term $\cos 2\pi t \times x$ represents population size multiplied by a factor which varies sinusoidally with time, and completes one cycle per year. Note that $\cos 2\pi t$ varies between $+1$ and -1, and is positive for half the year, and negative for the other half. So, $dx/dt = (k\cos 2\pi t)x$ gives a rate of change which varies with the seasons, as well as with population size.

(*iv*) Separating the equation,

$$\int \frac{1}{x}\,dx = \int k\cos 2\pi t\,dt \qquad \text{so} \qquad \ln x = \frac{k\sin 2\pi t}{2\pi} + C.$$

Therefore, $x = \exp\left(\dfrac{k\sin 2\pi t}{2\pi} + C\right)$.
Putting $x = x_0$ when $t = 0$ gives $x_0 = e^C$ so $x = x_0 e^{(k\sin 2\pi t)/2\pi}$.

(*v*) The exponent in this solution varies sinusoidally with time and so has no limiting value. The population size is cyclic, repeating itself each year.

(*vi*) The maximum value occurs when $\sin 2\pi t = 1$, and the minimum when $\sin 2\pi t = -1$. The values are $x = x_0 e^{k/2\pi}$ and $x = x_0 e^{-k/2\pi}$ respectively. They occur when $2\pi t = \pi/2$ or $t = 1/4$ and $2\pi t = 3\pi/2$ or $t = 3/4$, i.e. a quarter of a year and three quarters of a year later than the maximum of the growth rate (perhaps late summer and late winter).

More Questions

5. (*i*) Find the general solution of

$$\frac{dy}{dx} = \frac{\operatorname{cosec} y}{\sqrt{x^2+4}}\,.$$

(*ii*) Find the particular solution of

$$\frac{dx}{dt} = \frac{x^2}{\cos^2 t}$$

if $x = 1$ when $t = 0$.

Solution

(*i*) Separate the variables and integrate:

$$\int \frac{1}{\operatorname{cosec} y}dy = \int \frac{1}{\sqrt{x^2+4}}dx\,.$$

This simplifies if we write $1/\operatorname{cosec} y = \sin y$ and substitute $x = 2\sinh\theta$, $dx/d\theta = 2\cosh\theta$,

$$\int \sin y\,dy = \int \frac{2\cosh\theta\,d\theta}{\sqrt{4\cosh^2\theta}} = \int d\theta\,,$$

giving

$$-\cos y = \theta + C = \sinh^{-1}\frac{x}{2} + C\,.$$

(*ii*) Separate variables and integrate:

$$\begin{aligned}\int \frac{dx}{x^2} &= \int \frac{dt}{\cos^2 t}\\ &= \int \sec^2 t\,dt\end{aligned}$$

so

$$-\frac{1}{x} = \tan t + C.$$

When $t = 0$, $x = 1$ so $-1 = C$. Therefore the particular solution is

$$-\frac{1}{x} = \tan t - 1\,, \qquad \text{or} \qquad x = \frac{1}{1 - \tan t}\,.$$

6. Find the general solutions of all the separable equations in Question 1.

Solution

(*i*)

$$\int \frac{1}{y} dy = \int \frac{x^2}{(x^2+1)^{1/2}} dx\,.$$

The second integral has been found in Question 3 part (*ii*). So

$$\ln|y| = \frac{1}{2} x\sqrt{1+x^2} - \frac{1}{2}\sinh^{-1} x + C\,.$$

(*ii*)

$$\int e^{-y} dy = \int \frac{x^2}{(a^2-x^2)^{3/2}} dx\,.$$

The second integral has been found in Question 3 part (*i*). So

$$-e^{-y} = \frac{x}{\sqrt{a^2-x^2}} - \sin^{-1}\frac{x}{a} + C\,.$$

7. (*i*) Given $y = A\sqrt{x^2+1}$, where A is an arbitrary constant, show by substitution that it satisfies the differential equation $\dfrac{dy}{dx} = \dfrac{xy}{x^2+1}$.

(*ii*) Since the general solution of a first-order differential equation depends on one arbitrary constant, we see that the solution given in part (*i*) is the general solution of $\dfrac{dy}{dx} = \dfrac{xy}{x^2+1}$. Now find the particular solution satisfying the initial condition $y(0) = 1$.

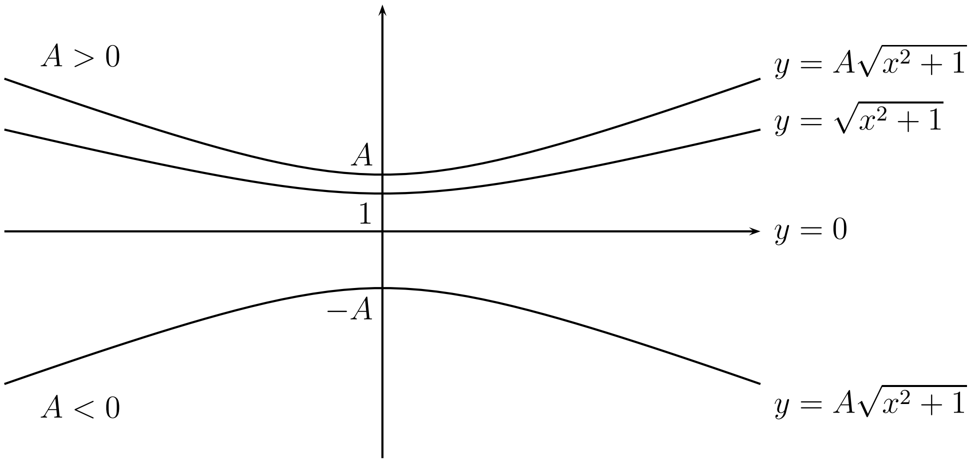
(*iii*) Sketch the family of solution curves given in part (*i*), indicating the behaviour of solutions for $A > 0$, $A = 0$ and $A < 0$. Indicate also on your sketch the particular solution found in part (*ii*).

Solution

(*i*) Differentiate $y = A\sqrt{x^2+1}$ to get LHS $= \dfrac{dy}{dx} = \dfrac{Ax}{\sqrt{x^2+1}}$. When we substitute y into the RHS we find $\dfrac{xy}{x^2+1} = \dfrac{Ax}{\sqrt{x^2+1}} =$ LHS, and so the given function satisfies the equation.

(*ii*) We require that $y(0) = 1$ and so must have $1 = A\sqrt{0^2+1} = A$. So $A = 1$ and the particular solution is $y = \sqrt{x^2+1}$.

(*iii*) $y = A\sqrt{x^2+1}$ so $y(0) = A$. $\dfrac{dy}{dx} = \dfrac{Ax}{\sqrt{x^2+1}}$ and so $\dfrac{dy}{dx} = 0$ when $x = 0$. For $A > 0$, $\dfrac{dy}{dx} < 0$ for $x < 0$ and $\dfrac{dy}{dx} > 0$ for $x > 0$. Then $x = 0$ is a global minimum. For $A < 0$, $\dfrac{dy}{dx} > 0$ for $x < 0$ and $\dfrac{dy}{dx} < 0$ for $x > 0$. Then $x = 0$ is a global maximum. Note that $\dfrac{d^2y}{dx^2} = \dfrac{A}{(x^2+1)^{3/2}}$, so when $A > 0$, $\dfrac{d^2y}{dx^2} > 0$, and when $A < 0$, $\dfrac{d^2y}{dx^2} < 0$. Note also that $A = 0$ gives the particular solution $y = 0$.



The function $y = A\sqrt{x^2+1}$ is an even function so its graph is symmetric about the y-axis.

Answers to Selected Questions

1. (*i*) Separable: $\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{x^2}{(x^2+1)^{1/2}}$. (*ii*) Separable: $e^{-y}\dfrac{dy}{dx} = \dfrac{x^2}{(a^2-x^2)^{3/2}}$.

(*iii*) Not separable.

2. (*i*) $y = \tan(x - C)$ (*ii*) $y = Ae^{\sin x}$ (*iii*) $y = \dfrac{1}{\ln|1+x| + C}$

3. (*i*) $\dfrac{x}{\sqrt{a^2-x^2}} - \sin^{-1}\dfrac{x}{a} + C$ (*ii*) $\dfrac{1}{2}x\sqrt{1+x^2} - \dfrac{1}{2}\sinh^{-1}x + C$

5. (*i*) $-\cos y = \sinh^{-1}\dfrac{x}{2} + C$ (*ii*) $x = \dfrac{1}{1-\tan t}$

6. (*i*) $\ln|y| = \dfrac{1}{2}x\sqrt{1+x^2} - \dfrac{1}{2}\sinh^{-1}x + C$ (*ii*) $-e^{-y} = \dfrac{x}{\sqrt{a^2-x^2}} - \sin^{-1}\dfrac{x}{a} + C$