THE UNIVERSITY OF SYDNEY
MATH1003 INTEGRAL CALCULUS AND MODELLING

Semester 2 **Exercises and Solutions for Week 10** 2009

Assumed Knowledge Integration techniques.

Objectives

(9a) To be able to distinguish between separable and linear first-order differential equations.

(9b) To be able to solve a linear equations utilising an integrating factor.

Preparatory Questions

1. For each of the following equations, determine whether it is separable or a first-order linear equation:

(i) $\dfrac{dy}{dx} + 3y = x$ $\qquad$ (ii) $t\dfrac{dx}{dt} + x = \cos t$

(iii) $\dfrac{x}{2}\dfrac{dy}{dx} = x^2 - y$ $\qquad$ (iv) $\dfrac{dy}{dx} = \dfrac{2x\sqrt{y}}{\sqrt{1+x^2}}.$

2. Write each of the linear equations in Question 1 in standard form $\dfrac{dy}{dx} + p(x)y = q(x)$ (with a suitable renaming of variables where necessary) and identify the functions p and q.

Practice Questions

3. (i) Find the general solution of $\dfrac{dy}{dx} + 3y = x.$

(ii) Find the general solution of $t\dfrac{dx}{dt} + x = \cos t$

(iii) Find the particular solution of $\dfrac{x}{2}\dfrac{dy}{dx} = x^2 - y$ for which $y = 1$ when $x = 1$.

Solution

(i) The integrating factor is $e^{\int 3dx} = e^{3x}$.
Multiply the equation by the integrating factor:

$$e^{3x}\frac{dy}{dx} + 3e^{3x}y = xe^{3x}$$
$$\frac{d}{dx}\left(e^{3x}y\right) = xe^{3x}\,.$$

Integrate both sides with respect to x.

$$\begin{aligned} e^{3x}y &= \int xe^{3x}\,dx \\ &= \frac{1}{3}xe^{3x} - \int \frac{1}{3}e^{3x}\,dx \qquad \text{(integration by parts)} \\ &= \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + C\,. \end{aligned}$$

Therefore the general solution is

$$y = \frac{1}{3}x - \frac{1}{9} + Ce^{-3x}.$$

(ii) Working with the standard form (see solutions to Preparatory Question), the integrating factor is $r(t) = e^{\int (1/t)dt} = e^{\ln t} = t$. Multiply the standard form by the integrating factor:

$$\begin{aligned} t\frac{dx}{dt} + x &= \cos t \\ \frac{d}{dt}(tx) &= \cos t\,. \end{aligned}$$

Integrate both sides with respect to t.

$$tx = \sin t + C\,.$$

So $x = \dfrac{\sin t}{t} + \dfrac{C}{t}$.

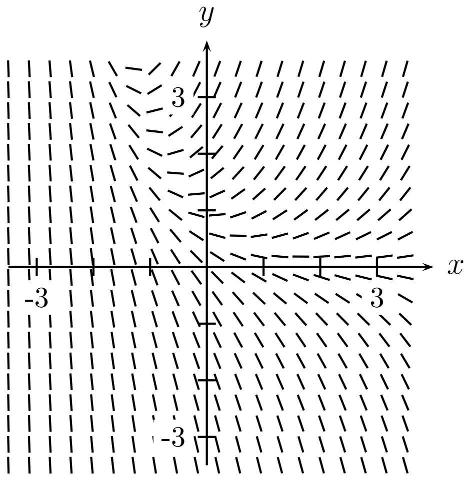
(iii) Working with the standard form (see solutions to Preparatory Question), the integrating factor is

$$r(x) = e^{\int (2/x)\,dx} = e^{2\ln x} = e^{\ln x^2} = x^2\,.$$

(We ignore the modulus in the log term as we only require one particular form for $r(x)$.)
Multiplying the standard form through by x^2 and rewriting the left-hand side of the DE produces $\dfrac{d}{dx}(x^2y) = 2x^3$, which integrates to $x^2y = \dfrac{x^4}{2} + C$, or $y = \dfrac{x^2}{2} + \dfrac{C}{x^2}$. Putting $y = 1$ when $x = 1$ shows that $C = \frac{1}{2}$, so that the required particular solution is $y = \dfrac{x^2}{2} + \dfrac{1}{2x^2}$.

4. (i) The direction field for the equation $\dfrac{dy}{dx} = y - e^{-x}$ is shown below.
Sketch the graphs of the solutions which satisfy the following initial conditions:

(a) $x = 0$, $y = 0$; (b) $x = 0$, $y = 1$.



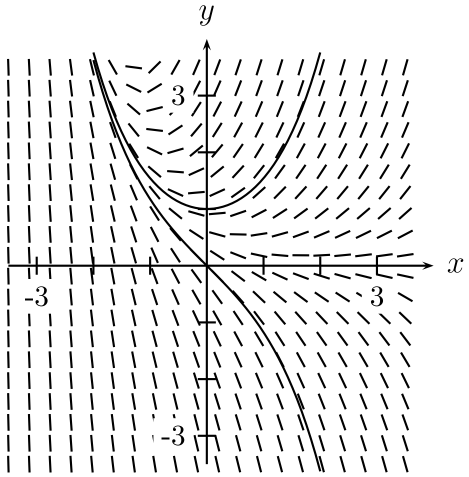
(ii) (a) Find the general solution to the above differential equation.

(b) Find the particular solutions which correspond to each of the curves sketched in part (i). What happens as $x \to \infty$ for each of these solutions?

(c) Find the particular solution for the initial condition $x = 0$, $y = \frac{1}{2}$. What happens as $x \to \infty$ in this case?

Solution

(i)



(ii) (a) This is a first-order linear equation.

In standard form it is $\dfrac{dy}{dx} - y = -e^{-x}$, with $p(x) = -1$. The integrating factor is therefore

$$r(x) = \exp\left(\int p(x)\, dx\right) = \exp\left(\int -1\, dx\right) = e^{-x}\,.$$

Multiplying through by e^{-x} and rewriting the left-hand side of the equation produces $\dfrac{d}{dx}(e^{-x}y) = -e^{-2x}$.

Integrating, we have $e^{-x}y = \dfrac{1}{2}e^{-2x} + C$, or

$$y = \frac{1}{2}e^{-x} + Ce^{x}.$$

(b) For the condition $y = 1$ when $x = 0$ we get $C + \frac{1}{2} = 1$ so $C = \frac{1}{2}$ and the required particular solution is

$$y = \frac{1}{2}e^{-x} + \frac{1}{2}e^{x} = \cosh x.$$

Note that as $x \to \infty$ the e^x term will dominate, and so this particular solution will tend to $+\infty$.
Putting $y = 0$ when $x = 0$ shows that $C + \frac{1}{2} = 0$ so $C = -\frac{1}{2}$ and the required particular solution is

$$y = \frac{1}{2}e^{-x} - \frac{1}{2}e^{x} = -\sinh x.$$

In this case, the e^x term will again dominate as $x \to \infty$ but now has a negative coefficient, and so this particular solution will tend to $-\infty$.

(c) If we have $y = \frac{1}{2}$ when $x = 0$ we get $C + \frac{1}{2} = \frac{1}{2}$, and so $C = 0$. In this case, the particular solution is $y = \frac{1}{2}e^{-x}$. This solution tends to 0 as $x \to \infty$ because the e^x term is absent. The case with $C = \frac{1}{2}$ is a special case which delineates between the cases where $y \to \infty$ and $y \to -\infty$ as $x \to \infty$.

5. (Suitable for group work and discussion.) A model for how the size of a fish varies with time is

$$\frac{dV}{dt} = -\alpha V + \beta S.$$

where V is the volume of the fish, S is its surface area, t a time variable and $\alpha > 0$ and $\beta > 0$ constants.

For particular species,

$$V = \frac{L^3}{10} \qquad \text{and} \qquad S = L^2\,,$$

and when L is measured in metres and time t in years the growth equation is

$$\frac{dV}{dt} = -V + \frac{S}{10}.$$

(i) Show that L satisfies the differential equation

$$\frac{dL}{dt} = \frac{1}{3}(1 - L)\,.$$

(ii) Solve this equation as a linear differential equation to find $L(t)$ given that $L = 0$ when $t = 0$.

(iii) What is the maximum size to which such a fish can grow?

(iv) How long does it take for such a fish to grow to 50 cm in length?

Solution

(i) Substitute $\dfrac{dV}{dt} = \dfrac{d}{dt}\left(\dfrac{L^3}{10}\right) = \dfrac{3L^2}{10}\dfrac{dL}{dt}$ to obtain

$$\begin{aligned}\frac{3L^2}{10}\frac{dL}{dt} &= -\frac{L^3}{10} + \frac{L^2}{10}\\ \frac{dL}{dt} &= \frac{1}{3}(-L + 1)\,.\end{aligned}$$

(*ii*) Writing this as

$$\frac{dL}{dt} + \frac{L}{3} = \frac{1}{3}$$

we find an integrating factor to be be $r(t) = e^{\int (1/3)\,dt} = e^{t/3}$. Multiplying through by this integrating factor produces

$$\frac{d}{dt}\left(e^{t/3}L\right) = \frac{e^{t/3}}{3}$$

which integrates to

$$e^{t/3}L = e^{t/3} + C\,.$$

Hence $L = 1 + Ce^{-t/3}$. Putting $L = 0$ when $t = 0$ requires $0 = 1 + C$, so $C = -1$. The length L as a function of time is thus

$$L = 1 - e^{-t/3}\,.$$

(*iii*) Clearly $L(t)$ is an increasing function of t, but as $t \to \infty$, $L \to 1$. So the maximum length is 1 metre.

(*iv*) The fish reaches 50 cm (0.5 m) when

$$0.5 = 1 - e^{-t/3}, \quad \text{or} \quad e^{-t/3} = \frac{1}{2}, \quad \text{or} \quad \frac{t}{3} = \ln 2,$$

i.e. after $t = 3\ln 2 = 2.08$ years.

More Questions

6. (*i*) For each of the following differential equations, find the general solution and also the particular solution satisfying $y(1) = 0$.

(a) $\dfrac{dy}{dx} + 4y = e^{-2x}$

(b) $\dfrac{dy}{dx} + (\sinh x)y = (2x)e^{-\cosh x}$

(*ii*) Find the general solution of the differential equation

$$\frac{dz}{dx} + (\cot x)z = -2x\,,$$

where we assume $0 < x < \pi$.

Solution

(*i*) (a) Integrating factor is $e^{\int 4\,dx} = e^{4x}$, and multiplying our equation by this gives

$$\frac{d}{dx}\left(e^{4x}y\right) = e^{2x}$$

and thus

$$e^{4x}y = \frac{1}{2}e^{2x} + C,$$

which then gives the general solution

$$y = \frac{1}{2}e^{-2x} + Ce^{-4x}.$$

The condition $y(1) = 0$ gives $0 = \frac{1}{2}e^{-2} + Ce^{-4}$, and so $C = -\frac{1}{2}e^2$. Thus the required particular solution is

$$y = \frac{1}{2}e^{-2x}\left(1 - e^{2(1-x)}\right).$$

(b) Integrating factor is $e^{\int \sinh x\, dx} = e^{\cosh x}$. Multiplying our equation by this integrating factor gives

$$\frac{d}{dx}\left(e^{\cosh x}y\right) = 2x,$$

and so

$$e^{\cosh x}y = x^2 + C,$$

which then gives the general solution

$$y = (x^2 + C)e^{-\cosh x}.$$

The condition $y(1) = 0$ means $0 = (1 + C)e^{(-\cosh(1))}$, and so $C = -1$. Thus the required particular solution is

$$y = (x^2 - 1)e^{-\cosh x}.$$

(*ii*) This is a first-order linear equation. Its integrating factor is

$$e^{\int \cot x\, dx} = e^{\int \frac{\cos x}{\sin x}\, dx} = e^{\ln(\sin x)} = \sin x.$$

Multiplying our equation by this integrating factor gives

$$(\sin x)\frac{dz}{dx} + (\cos x)z = -2x\sin x$$

i.e.

$$\frac{d}{dt}(z\sin x) = -2x\sin x.$$

We then integrate by parts to get

$$z\sin x = 2x\cos x - \int 2\cos x\, dx = 2x\cos x - 2\sin x + C,$$

and thus we obtain the general solution as

$$z = \frac{2x\cos x - 2\sin x + C}{\sin x}.$$

7. Consider the equation

$$\frac{dy}{dx} + y\cos x = \cos x.$$

Solve this equation as a linear equation and then solve it as a separable equation. Are the solutions the same?

Solution The equation is already in standard first-order linear form with $p(x) = \cos x$. So the integrating factor is $\exp\left(\int \cos x dx\right) = e^{\sin x}$. Thus

$$\frac{d}{dx}\left(ye^{\sin x}\right) = \cos x\, e^{\sin x}$$
$$ye^{\sin x} = e^{\sin x} + C$$
$$y = 1 + Ce^{-\sin x}.$$

Solving by separating the variables, we have

$$\begin{aligned}
\int \frac{1}{1-y}dy &= \int \cos x\,dx \\
-\ln|1-y| &= \sin x + C \\
\ln|1-y| &= -\sin x - C \\
|1-y| &= e^{-C}e^{-\sin x} \\
1-y &= Ae^{-\sin x} \\
y &= 1-Ae^{-\sin x}\,.
\end{aligned}$$

The way that the constants of integration occur in the two solution methods is slightly different but the solutions are, of course, the same. To see this, simply replace A with $-C$ in the solution above.

8. In electronic circuit theory, circuits with a resistor and an inductance coil in series with a voltage applied across these two components are known as RL circuits. This is because the resistance of the resistor is conventionally given as R ohms and the inductance of the coil is conventionally given as L henries. The equation for the rate of change of the electric current I in such a circuit is

$$\frac{dI}{dt} + \frac{R}{L}I = \frac{V}{L}$$

where V is the voltage applied to the circuit. In a circuit with an applied AC current, V will vary with time as $V = A\sin\omega t$. So, if R and L are constant the equation becomes

$$\frac{dI}{dt} + \frac{R}{L}I = \frac{A\sin\omega t}{L}.$$

Solve this equation to find the general solution for I as a function of t. Find the particular solution if the circuit has no current in it when it is switched on. What happens to the current as $t \to \infty$? How does the initial condition affect this long-term behaviour?

(Hint: $\int e^{au}\sin bu\,du = \frac{e^{au}}{a^2+b^2}(a\sin bu - b\cos bu) + C$.)

Solution Equation is first-order linear with $p(t) = R/L$. So the integrating fractor is

$$r(t) = \exp\left(\int \frac{R}{L}dt\right) = e^{Rt/L}.$$

So the differential equation becomes

$$\frac{d}{dt}\left(e^{Rt/L}I\right) = \frac{A\sin\omega t}{L}e^{Rt/L},$$

since R and L are constants. Integrating both sides with respect to t:

$$\begin{aligned}
e^{Rt/L}I &= \frac{A}{L}\frac{e^{Rt/L}}{(R/L)^2+\omega^2}\left(\frac{R}{L}\sin\omega t - \omega\cos\omega t\right) + C \\
I &= \frac{A}{L}\frac{1}{(R/L)^2+\omega^2}\left(\frac{R}{L}\sin\omega t - \omega\cos\omega t\right) + Ce^{-Rt/L}.
\end{aligned}$$

This is the general solution. To find the particular solution let $I = 0$ when $t = 0$. Then the equation gives

$$\frac{A}{L}\frac{-\omega}{(R/L)^2+\omega^2} + C = 0$$

so

$$C = \frac{A}{L}\frac{\omega}{(R/L)^2+\omega^2}$$

and the particular solution is

$$I = \frac{A}{L}\frac{1}{(R/L)^2+\omega^2}\left(\frac{R}{L}\sin\omega t - \omega\cos\omega t + \omega e^{-Rt/L}\right).$$

As $t \to \infty$ $e^{-Rt/L} \to 0$ leaving just the sine and cosine terms. So eventually the current will be a periodic function of time (that is it will oscillate) with a period of $2\pi/\omega$. The initial condition only contributed to the $e^{-Rt/L}$ term (the transient term). Hence it has no effect on the long term behaviour.

Answers to Selected Questions

1. Equations (i), (ii) and (iii) are linear, (iv) is separable.

2. (i) The equation is in standard form, with $p(x) = 3$, and $q(x) = x$.

(ii) Standard form $\dfrac{dx}{dt} + \dfrac{x}{t} = \dfrac{\cos t}{t}$. So solution is $p(t) = \dfrac{1}{t}$ and $q(t) = \dfrac{\cos t}{t}$.

(iii) Standard form $\dfrac{dy}{dx} + \dfrac{2y}{x} = 2x$. Solution $p(x) = \dfrac{2}{x}$ and $q(x) = 2x$.

3. (i) $y = \frac{1}{3}x - \frac{1}{9} + Ce^{-3x}$ (ii) $x = \dfrac{\sin t}{t} + \dfrac{C}{t}$ (iii) $y = \dfrac{x^2}{2} + \dfrac{1}{2x^2}$

4. (ii) (a) $y = \frac{1}{2}e^{-x} + Ce^{x}$

(b) For $y = 1$, $x = 0$ particular solution is $y = \frac{1}{2}e^{-x} + \frac{1}{2}e^{x} = \cosh x$.

For $y = 0$, $x = 0$ particular solution is $y = \frac{1}{2}e^{-x} - \frac{1}{2}e^{x} = -\sinh x$.

5. (i) $L = 1 - e^{-t/3}$ (ii) 1 metre (iii) $t = 3\ln 2 = 2.08$ years

6. (i) (a) General solution: $y = \frac{1}{2}e^{-2x} + Ce^{-4x}$, particular solution: $y = \frac{1}{2}e^{-2x}\left(1 - e^{2(1-x)}\right)$.

(b) General solution: $y = (x^2 + C)e^{-\cosh x}$, particular solution: $y = (x^2 - 1)e^{-\cosh x}$.

(ii) $z = \dfrac{2x\cos x - 2\sin x + C}{\sin x}$

8. General solution:

$$I = \frac{A}{L}\frac{1}{(R/L)^2+\omega^2}\left(\frac{R}{L}\sin\omega t - \omega\cos\omega t\right) + Ce^{-Rt/L}$$

Particular solution:

$$I = \frac{A}{L}\frac{1}{(R/L)^2+\omega^2}\left(\frac{R}{L}\sin\omega t - \omega\cos\omega t + \omega e^{-Rt/L}\right).$$

As $t \to \infty$

$$I \to \frac{A}{L}\frac{1}{(R/L)^2+\omega^2}\left(\frac{R}{L}\sin\omega t - \omega\cos\omega t\right).$$