

Assumed Knowledge Integration techniques.

Objectives

- (10a) To be able to solve differential equations that are separable, linear or both.
(10b) To be able to construct and solve equations describing flow and mixing problems.

Preparatory Questions

1. Classify each of the following differential equations:

(i) $(1 + x^2) \frac{dy}{dx} = 1 - 2xy$

(ii) $(1 + x^2) \frac{dy}{dx} = y - 2xy$

Practice Questions

2. Find the general solutions of the differential equations in Preparatory Question 1.
3. In a prolific breed of rabbits the rate of birth and the rate of death are each proportional to the square of the population N . Let t be a time variable and k_1, k_2 be the constants of proportionality for the rates of births and deaths respectively. Assume $k_1 > k_2$ and write $k = k_1 - k_2$. Then $dN/dt = kN^2$, where $k > 0$. Solve this differential equation to show that

$$N(t) = \frac{N(0)}{1 - kN(0)t},$$

where $N(0)$ is the initial population.

Suppose we start with an initial population of 2 rabbits and that there are 4 rabbits after 3 months. What does this model predict will happen after another 3 months?

4. (Suitable for group work and discussion.) When people smoke, carbon monoxide is released into the air. In a room of volume 50 m^3 , smokers introduce air containing 0.05 mg/m^3 of carbon monoxide at the rate of $0.002 \text{ m}^3/\text{min}$. Assume that the smoky air mixes immediately with the rest of the air, and that the mixture is pumped through an air purifier at a rate of $0.002 \text{ m}^3/\text{min}$. The purifier removes all the carbon monoxide from the air passing through it.
- (i) Write a differential equation for $m(t)$, the mass of carbon monoxide in the room at time t , where t is measured in minutes.
(ii) Solve the differential equation, assuming that there was no carbon monoxide in the room initially.

- (iii) What happens to the value of $m(t)$ in the long run?
- (iv) It is dangerous for people to be in the room if the mass of carbon monoxide per unit volume reaches 0.001 mg/m^3 . How long does it take for this to happen?

More Questions

5. Radiocarbon dating allows us to estimate the age of ancient objects. In living organisms the ratio of radioactive carbon-14 to ordinary carbon-12 is constant. However, when the organism dies, carbon-14 is no longer absorbed, from the atmosphere or via feeding for example, and so the amount present decreases through radioactive decay. By comparing the amount of carbon-14 present with the amount which would normally be present, we can determine the number of years since an organism died (or the age of objects such as clothing or paper, made from once-living material). The half-life of carbon-14 is 5730 years.

If y is the units of mass of carbon-14 and t a time variable

$$\frac{dy}{dt} = -ky, \quad (k > 0).$$

The half life tells us that half of any given mass of carbon-14 will disintegrate in 5730 years.

- (i) Find the solution of this equation given that at time $t = 0$, the amount of carbon-14 present is y_0 .
 - (ii) Use the fact that half of this amount y_0 will disintegrate over 5730 years to find a numerical value for k when time t is measured in years.
 - (iii) A piece of woollen clothing is found to have only 77% of the amount of carbon-14 normally found in wool. Estimate the age of this piece of clothing.
6. A lucerne crop on an experimental farm is grown on an unirrigated paddock. The rate of growth of the crop depends on the mass of the lucerne plants and the water content of the soil. The water content of soil is observed to decline exponentially with time when there is no rain. Once the soil water content falls below a critical level the crop stops growing and starts to die back.

- (i) The following equation is proposed as a model for this crop's growth:

$$\frac{ds}{dt} = m(M - W_c)s, \quad (m > 0 \text{ constant})$$

where s is the mass per unit area of the crop, t is time, $M = M(t)$ the soil moisture content, and W_c the critical soil moisture content for the crop.

Explain why this equation might be a reasonable model for crop growth.

- (ii) In the absence of rain $M = Ke^{-at}$, $K > 0$ and $a > 0$ constants. Sketch M as a function of t .
- (iii) Find the time t when the crop stops growing and starts to die back.
- (iv) Find the general solution to the differential equation in part (i), with M as in (ii).

- (v) An agricultural scientist has data on this crop in this paddock from the previous seasons. He knows that for s in kilograms per hectare and t in weeks, $m = 0.035$, $W_c = 1.3$ and $a = 0.41$.
- A crop is planted and starts to grow. Then heavy rain falls and increases the soil moisture content to 15 times W_c , the critical level. Let $t = 0$ immediately after the rain. Use the expression for soil moisture $M(t)$ to find K .
 - How soon after the rain does the crop stop growing?
 - When the rain falls the crop has grown to a mass of 400 kg per hectare. What is the maximum mass per hectare attained by the crop if there is no further rain? If the crop is cut 12 weeks after the rain what will be the yield per hectare?
7. A holding pond at a chemical treatment plant contains 2000 litres of water which has been contaminated with a particular impurity. The concentration of this impurity in the pond is initially 5 grams per litre. Water is drained from the pond for treatment at a rate of 10 litres per minute and water from another pond is added at the same rate. This input water also contains the impurity at a concentration of 0.02 grams per litre. A safe level of this impurity is considered to be 1 gram per litre. Formulate a differential equation for the rate of change of the mass of impurity in the holding pond. Assuming that the water in the holding pond is well-stirred, how long does it take for the impurity to reach safe levels ?

Answers to Selected Questions

- Linear.
 - Both linear and separable.
- $y = \frac{x + C}{1 + x^2}$
 - $y = \frac{C e^{(\tan^{-1} x)}}{1 + x^2}$
- $\frac{dm}{dt} = 0.00002(5 - 2m)$
 - $m = \frac{5}{2} \left(1 - e^{-0.00004t} \right)$
 - $m \rightarrow \frac{5}{2}$
 - $t \approx 500$ minutes
- $y = y_0 e^{-kt}$
 - $k = 0.000121$
 - $t \approx 2160$ years
- $t = \frac{1}{a} \ln(K/W_c)$
 - $s = A \exp \left\{ m \left(\frac{-K}{a} e^{-at} - W_c t \right) \right\}$
 - (v) (a) $K = 19.5$ (b) 6.6 weeks after the rain. (c) Maximum biomass occurs after 6.6 weeks and is 1398 kg per hectare. (d) 1210 kg per hectare.
- $\frac{dI}{dt} = -0.005I + 0.2$. Hence $I = 200(0.2 + 49.8e^{-0.005t})$. Impurity reaches safe levels after 5 hours and 25 minutes.