The University of Sydney
Math1003 Integral Calculus and Modelling

Semester 2 **Exercises and Solutions for Week 11** 2009

Assumed Knowledge Integration techniques.

Objectives

(10a) To be able to solve differential equations that are separable, linear or both.

(10b) To be able to construct and solve equations describing flow and mixing problems.

Preparatory Questions

1. Classify each of the following differential equations:

(i) $(1+x^2)\dfrac{dy}{dx} = 1 - 2xy$ (ii) $(1+x^2)\dfrac{dy}{dx} = y - 2xy$

Practice Questions

2. Find the general solutions of the differential equations in Preparatory Question 1.

Solution

(i) This linear equation has standard form

$$\frac{dy}{dx} + \frac{2x}{1+x^2}y = \frac{1}{1+x^2}\,.$$

Multiplying by the integrating factor

$$\exp\left(\int \frac{2x}{1+x^2}\,dx\right) = \exp(\ln(1+x^2)) = 1+x^2\,,$$

gives

$$\begin{aligned}(1+x^2)\frac{dy}{dx} + 2xy &= 1\\ \frac{d}{dx}\big((1+x^2)y\big) &= 1\\ (1+x^2)y &= x + C\\ y &= \frac{x+C}{1+x^2}\end{aligned}$$

(ii) This equation is both separable and linear.

Solution separating variables:

$$\begin{aligned}
(1+x^2)\frac{dy}{dx} &= y - 2xy = y(1-2x)\\
\frac{dy}{dx} &= \left(\frac{1-2x}{1+x^2}\right)y\\
\frac{1}{y}\,dy &= \frac{1-2x}{1+x^2}\,dx\\
\int \frac{1}{y}\,dy &= \int \frac{1-2x}{1+x^2}\,dx = \int \left(\frac{1}{1+x^2} - \frac{2x}{1+x^2}\right)dx\\
\ln|y| &= \tan^{-1}x - \ln(1+x^2) + C\\
y &= A\exp\left(\tan^{-1}x + \ln\left(\frac{1}{1+x^2}\right)\right) = \frac{Ae^{(\tan^{-1}x)}}{1+x^2}\,.
\end{aligned}$$

Solution as a linear equation. Multiplying its standard form

$$\frac{dy}{dx} + \left(\frac{2x-1}{1+x^2}\right)y = 0\,,$$

by the integrating factor

$$\exp\left(\int \frac{2x-1}{1+x^2}\,dx\right) = \exp(\ln(1+x^2) - \tan^{-1}x) = (1+x^2)e^{(-\tan^{-1}x)}.$$

we find

$$\frac{d}{dx}\left((1+x^2)e^{(-\tan^{-1}x)}y\right) = 0, \quad (1+x^2)e^{(-\tan^{-1}x)}y = C, \text{ and finally } \quad y = \frac{Ce^{(\tan^{-1}x)}}{1+x^2}\,.$$

3. In a prolific breed of rabbits the rate of birth and the rate of death are each proportional to the square of the population N. Let t be a time variable and k_1, k_2 be the constants of proportionality for the rates of births and deaths respectively. Assume $k_1 > k_2$ and write $k = k_1 - k_2$. Then $dN/dt = kN^2$, where $k > 0$. Solve this differential equation to show that

$$N(t) = \frac{N(0)}{1 - kN(0)t}\,,$$

where $N(0)$ is the initial population.

Suppose we start with an initial population of 2 rabbits and that there are 4 rabbits after 3 months. What does this model predict will happen after another 3 months?

Solution The differential equation can be solving by separating and integrating

$$\int \frac{1}{N^2}\,dN = k\int dt, \qquad -\frac{1}{N} = kt + C\,.$$

Putting $t = 0$ gives $C = -1/N(0)$.

Substituting and rearranging gives the required solution.

Suppose the time variable t is elapsed months. We are given that $N(0) = 2$, and $N(3) = 4$. Substitution then shows that $4 = \dfrac{2}{1-6k}$, and solving gives $k = 1/12$.

Thus the population as a function of time in this case is $N = \dfrac{2}{1-t/6}$. As we head for 6 months the rabbit population heads to infinity. Only half that time earlier the population was just 4. At large N this simple model is no longer adequate.

4. (Suitable for group work and discussion.) When people smoke, carbon monoxide is released into the air. In a room of volume $50\,\mathrm{m}^3$, smokers introduce air containing $0.05\,\mathrm{mg/m}^3$ of carbon monoxide at the rate of $0.002\,\mathrm{m}^3/\mathrm{min}$. Assume that the smoky air mixes immediately with the rest of the air, and that the mixture is pumped through an air purifier at a rate of $0.002\,\mathrm{m}^3/\mathrm{min}$. The purifier removes all the carbon monoxide from the air passing through it.

(*i*) Write a differential equation for $m(t)$, the mass of carbon monoxide in the room at time t, where t is measured in minutes.

(*ii*) Solve the differential equation, assuming that there was no carbon monoxide in the room initially.

(*iii*) What happens to the value of $m(t)$ in the long run?

(*iv*) It is dangerous for people to be in the room if the mass of carbon monoxide per unit volume reaches $0.001\,\mathrm{mg/m}^3$. How long does it take for this to happen?

Solution

(*i*) Let m be the milligrams of carbon monoxide in the room and minutes t elapsed since smoking started.

$$\text{room CO mass accumulation rate} = \text{CO mass rate in} - \text{CO mass rate out}.$$

The smoky air is entering at the rate of $0.002\,\mathrm{m}^3/\mathrm{min}$ and contains $0.05\,\mathrm{mg/m}^3$ of CO. So the rate in is $0.05 \times 0.002\,\mathrm{mg/min}$. At time t, the $50\,\mathrm{m}^3$ room contains m mg of CO, and so the concentration of CO is $\frac{m}{50}\,\mathrm{mg/m}^3$ and the rate at which it is removed is $0.002 \times \frac{m}{50}\,\mathrm{mg/min}$. Therefore

$$\frac{dm}{dt} = 0.05 \times 0.002 - 0.002 \times \frac{m}{50} = 0.00002(5 - 2m).$$

(*ii*) This is a separable equation:

$$\begin{aligned} \int \frac{dm}{5-2m} &= 0.00002 \int dt \\ -\frac{1}{2}\ln|5-2m| &= 0.00002t + C \\ 5 - 2m &= Ae^{-0.00004t} \qquad (A = \pm e^{2C}). \end{aligned}$$

Since there is no carbon monoxide in the room initially, $m = 0$ when $t = 0$, and so $A = 5$. We therefore have

$$m = \frac{5}{2}\left(1 - e^{-0.00004t}\right).$$

(*iii*) In the long run $m \to 5/2$. That is, in the $50\,\mathrm{m}^3$ room, the concentration of CO is eventually $2.5/50 = 0.05\,\mathrm{mg/m}^3$.

(*iv*) At the time when the mass of CO is $0.001 \times 50 = 0.05\,\mathrm{mg}$

$$0.05 = \frac{5}{2}\left(1 - e^{-0.00004t}\right).$$

Thus happens after time $t = -\dfrac{\ln(0.98)}{0.00004} \approx 500\,\mathrm{min}$.

More Questions

5. Radiocarbon dating allows us to estimate the age of ancient objects. In living organisms the ratio of radioactive carbon-14 to ordinary carbon-12 is constant. However, when the organism dies, carbon-14 is no longer absorbed, from the atmosphere or via feeding for example, and so the amount present decreases through radioactive decay. By comparing the amount of carbon-14 present with the amount which would normally be present, we can determine the number of years since an organism died (or the age of objects such as clothing or paper, made from once-living material). The half-life of carbon-14 is 5730 years.

If y is the units of mass of carbon-14 and t a time variable

$$\frac{dy}{dt} = -ky, \qquad (k > 0).$$

The half life tells us that half of any given mass of carbon-14 will disintegrate in 5730 years.

(i) Find the solution of this equation given that at time $t = 0$, the amount of carbon-14 present is y_0.

(ii) Use the fact that half of this amount y_0 will disintegrate over 5730 years to find a numerical value for k when time t is measured in years.

(iii) A piece of woollen clothing is found to have only 77% of the amount of carbon-14 normally found in wool. Estimate the age of this piece of clothing.

Solution

(i) Exponential decay equations are both separable and linear with standard form

$$\frac{dy}{dx} + ky = 0.$$

Multiplying by integrating factor $e^{\int k\,dt} = e^{kt}$, and integrating gives first $\frac{d}{dt}\left(e^{kt}y\right) = 0$, and then $e^{kt}y = C$.
Putting $t = 0$, gives $C = y_0$, and finally $y = y_0e^{-kt}$.

(ii) We have $y/y_0 = e^{-kt}$ and when t is measured in years $e^{-5730k} = 1/2$.
Solving: $k = (-1/5730)\ln(\frac{1}{2}) = 0.000121$ (to 3 significant figures).

(iii) If 77% of the original amount of carbon-14 remains, after t years then $e^{-0.000121t} = y/y_0 = 0.77$.
This gives $t = (-1/0.000121)\ln(0.77) \approx 2160$ years as the age of the clothing.

6. A lucerne crop on an experimental farm is grown on an unirrigated paddock. The rate of growth of the crop depends on the mass of the lucerne plants and the water content of the soil. The water content of soil is observed to decline exponentially with time when there is no rain. Once the soil water content falls below a critical level the crop stops growing and starts to die back.

(i) The following equation is proposed as a model for this crop's growth:

$$\frac{ds}{dt} = m(M - W_c)s, \qquad (m > 0 \text{ constant})$$

where s is the mass per unit area of the crop, t is time, $M = M(t)$ the soil moisture content, and W_c the critical soil moisture content for the crop.

Explain why this equation might be a reasonable model for crop growth.

(*ii*) In the absence of rain $M = Ke^{-at}$, $K > 0$ and $a > 0$ constants.
Sketch M as a function of t.

(*iii*) Find the time t when the crop stops growing and starts to die back.

(*iv*) Find the general solution to the differential equation in part (*i*), with M as in (*ii*).

(*v*) An agricultural scientist has data on this crop in this paddock from the previous seasons. He knows that for s in kilograms per hectare and t in weeks, $m = 0.035$, $W_c = 1.3$ and $a = 0.41$.

(a) A crop is planted and starts to grow. Then heavy rain falls and increases the soil moisture content to 15 times W_c, the critical level. Let $t = 0$ immediately after the rain. Use the expression for soil moisture $M(t)$ to find K.

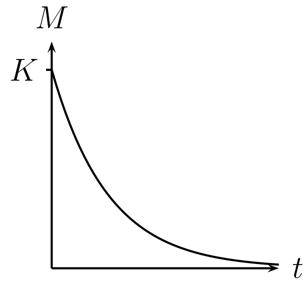
(b) How soon after the rain does the crop stop growing?

(c) When the rain falls the crop has grown to a mass of 400 kg per hectare. What is the maximum mass per hectare attained by the crop if there is no further rain? If the crop is cut 12 weeks after the rain what will be the yield per hectare?

Solution

(*i*) This equation assumes the rate of change $\frac{ds}{dt}$ is proportional to crop size s; and to $(M(t) - W_c)$, the amount of soil moisture above the critical level W_c and $(m > 0)$ that $\frac{ds}{dt}$ is positive exactly when $(M(t) - W_c)$ is positive. This fits all the known (important) facts about the crop.

(*ii*)



(*iii*) When the crop stops growing $\frac{ds}{dt} = 0$. So $m(Ke^{-at} - W_c) = 0$, or $Ke^{-at} - W_c = 0$.
Solving for t yields $t = \frac{1}{a}\ln(K/W_c)$.

(*iv*) Separating the variables and integrating yields

$$\int \frac{ds}{s} = \int m(Ke^{-at} - W_c)dt, \quad \text{then} \quad \ln|s| = m\left(\frac{-K}{a}e^{-at} - W_c t\right) + C,$$

where C is a constant. This gives general solution

$$s = A\exp\left\{m\left(\frac{-K}{a}e^{-at} - W_c t\right)\right\}, \quad A = \pm e^C .$$

(*v*) (a) When $t = 0$, $M = 15W_c$; so $15W_c = Ke^0$. Hence $K = 15W_c = 19.5$.

(b) The crop stops growing when $t = 1/a \ln(K/W_c)$ (from part (*iii*)); that is, when $t = 6.6$. The crop stops growing 6.6 weeks after the rain.

(c) If $s = 400$ when $t = 0$ then

$$400 = A \exp\left(\frac{-mK}{a}\right), \quad \text{or} \quad A = 400 \exp\left(\frac{mK}{a}\right) \approx 2113\,.$$

Therefore after t weeks there are

$$s = 2113 \exp\left(0.035\left(-47.5e^{-0.41t} - 1.3t\right)\right), \quad \text{Kg per hectacre.}$$

The mass per hectare is at a maximum when $t = 6.6$. Substituting this into the equation for s we get $s = 1398$. So the crop attains a maximum of 1400 kg per hectare (correct to 3 significant figures).
When $t = 12$, $s = 1210$. So 12 weeks after the rain, when the crop is cut, the yield will be 1210 kg per hectare.

7. A holding pond at a chemical treatment plant contains 2000 litres of water which has been contaminated with a particular impurity. The concentration of this impurity in the pond is initially 5 grams per litre. Water is drained from the pond for treatment at a rate of 10 litres per minute and water from another pond is added at the same rate. This input water also contains the impurity at a concentration of 0.02 grams per litre. A safe level of this impurity is considered to be 1 gram per litre. Formulate a differential equation for the rate of change of the mass of impurity in the holding pond. Assuming that the water in the holding pond is well-stirred, how long does it take for the impurity to reach safe levels ?

Solution Let I be the mass in grams of impurity in the pond as function of elapsed minutes t. Because inflow and outflow rates are equal, the volume of water in the tank remains constant at 2000 litres.
Now, rate of loss of impurity through draining = $10I/2000$ grams/minute $=0.005I$.
Rate of input of impurity through input water is $10 \times 0.02 = 0.2$ grams per minute.
So

$$\frac{dI}{dt} = -\frac{10I}{2000} + 0.2 = -0.005I + 0.2\,.$$

Separating, integrating and solving for I: $\displaystyle\int dt = \int \frac{dI}{-0.005I + 0.2}$,

$$t + C = \frac{-1}{0.005} \ln|0.2 - 0.005I|, \quad \text{and finally} \quad I = 200(0.2 - Ae^{-0.005t})\,.$$

When $t = 0$, $I = 5 \times 2000 = 10,000$. So $10000 = 200(0.2 - A) \Longrightarrow A = -49.8$.
Hence $I = 200(0.2 + 49.8e^{-0.005t})$.
The impurity reaches safe levels in the pond when $I = 2000$ grams. This is when $(0.2 + 49.8e^{-0.005t}) = 2000/200 = 10$, which yields $t = (\ln(9.8/49.8))/(-0.005) \approx 325$.

The impurities falls to a safe level after $t = 325$ minutes i.e 5 hours 25 minutes.

Answers to Preparatory Questions

1. (*i*) Linear. (*ii*) Both linear and separable.