THE UNIVERSITY OF SYDNEY
MATH1003 INTEGRAL CALCULUS AND MODELLING

Semester 2 **Exercises and Solutions for Week 12** 2009

Assumed Knowledge Finding the roots of quadratic equations.

Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$.

Objectives

(11a) To be able to write down the auxiliary (characteristic) equation associated with a second-order differential equation with constant coefficients.

(11b) To be able to construct the solutions to such differential equations in terms of real exponential and trigonometric functions.

Preparatory Questions

1. Write down the auxiliary (characteristic) equations for each the following second-order linear differential equations with constant coefficients, and find their roots:

(*i*) $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} - 8y = 0$ (*ii*) $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} - 4y = 0$

(*iii*) $\dfrac{d^2y}{dt^2} - 9y = 0$ (*iv*) $\dfrac{d^2y}{dt^2} - 2\dfrac{dy}{dt} + 5y = 0$

(*v*) $\dfrac{d^2x}{dt^2} + 2\dfrac{dx}{dt} + x = 0$

Practice Questions

2. Find the particular solution of Preparatory Question 1 (*i*) with $y(0) = 0$ and $y'(0) = 3$.

Solution The roots of the auxiliary equation are $\lambda = 2$ and $\lambda = -4$.
Therefore, the general solution is $y = Ae^{2t} + Be^{-4t}$.
The condition $y(0) = 0$, gives $A + B = 0$ (and so $B = -A$).
Now, $y'(t) = 2Ae^{2t} - 4Be^{-4t}$.
The condition $y'(0) = 3$ gives $2A - 4B = 6A$. Therefore, $A = \frac{1}{2}$ and $B = -\frac{1}{2}$.
The particular solution is $y = \frac{1}{2}e^{2t} - \frac{1}{2}e^{-4t}$.

3. Find the general solution of Preparatory Question 1 (*ii*).

Solution The roots of the auxiliary equation are $\lambda = -1 + \sqrt{5}$ and $\lambda = -1 - \sqrt{5}$.
Hence $y = Ae^{(-1+\sqrt{5})t} + Be^{(-1-\sqrt{5})t} = e^{-t}(Ae^{\sqrt{5}t} + Be^{-\sqrt{5}t})$.

4. Find the particular solution of Preparatory Question 1 (*iii*) which satisfies the initial conditions $y = 3$ and $\dfrac{dy}{dt} = 3$ when $t = 0$.

Solution The roots of the auxiliary equation are $\lambda = 3$ and $\lambda = -3$.
The general solution is therefore $y = Ae^{3t} + Be^{-3t}$.

For this general solution $dy/dt = 3Ae^{3t} - 3Be^{-3t}$.
Putting $t = 0$ the initial conditions become $A + B = 3$ and $3A - 3B = 3$.
Solving these equations gives $A = 2$ and $B = 1$.
So the particular solution is $y = 2e^{3t} + e^{-3t}$.

5. Find the general solution of Preparatory Question 1 (*iv*).
Express your answer in terms of real functions.
What is the particular solution satisfying $y(0) = 1$ and $y(\pi/4) = 2$?

Solution The roots of the auxiliary equation are $\lambda = \dfrac{2 \pm \sqrt{4 - 20}}{2} = 1 \pm 2i$.
The general solution of the differential equation is therefore

$$y = Ae^{(1+2i)t} + Be^{(1-2i)t} = e^t(Ae^{2it} + Be^{-2it}).$$

In terms of real functions, the general solution is

$$y = e^t \left(C \cos 2t + D \sin 2t\right).$$

The condition $y(0) = 1$, gives $C = 1$.
The condition $y(\pi/4) = 2$, gives $2 = e^{\pi/4}D$, i.e. $D = 2e^{-\pi/4}$.
The required particular solution is therefore $y = e^t(\cos 2t + 2e^{-\pi/4} \sin 2t)$.

6. Find the particular solution of Preparatory Question 1 (*v*) which satisfies the initial conditions $x(0) = 1$ and $x'(0) = 2$.

Solution The auxiliary equation has two equal roots, $\lambda = -1$.
The general solution to the differential equation is

$$x = Ate^{-t} + Be^{-t}.$$

The condition $x(0) = 1$ gives $B = 1$ and $x = Ate^{-t} + e^{-t}$.
Now, $x'(t) = A(-te^{-t} + e^{-t}) - e^{-t}$. So $x'(0) = 2$ gives $A - 1 = 2$ and $A = 3$.
The particular solution is therefore $x = 3te^{-t} + e^{-t}$.

More Questions

7. Find the general solutions of these second-order homogeneous equations.
In each case give your answer in terms of real functions.

(*i*) $2\dfrac{d^2y}{dx^2} - 7\dfrac{dy}{dx} + 3y = 0.$ (*ii*) $\dfrac{d^2y}{dx^2} + 3y = 0.$

(*iii*) $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 2y = 0.$

Solution

(*i*) The auxiliary equation is $2\lambda^2 - 7\lambda + 3 = 0$. Roots $\lambda = \frac{1}{2}, 3$.
General solution is $y = Ae^{x/2} + Be^{3x}$.

(*ii*) The auxiliary equation is $\lambda^2 + 3 = 0$. Roots $\lambda = \pm i\sqrt{3}$.
General solution is $y = A \cos \sqrt{3}x + B \sin \sqrt{3}x$.

(*iii*) The auxiliary equation is $\lambda^2 - 2\lambda + 2 = 0$. Roots $\lambda = \dfrac{2 \pm \sqrt{4 - 8}}{2} = 1 \pm i$.
Therefore real form of the general solution is $y = e^x(A \cos x + B \sin x)$.

8. Find the general solution of each of these second-order linear homogeneous equations. Hence find the particular solution for the given conditions.
In each case give your answer in terms of real functions.

(*i*) $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 20y = 0, \quad y(0) = y'(0) = 1$

(*ii*) $\dfrac{d^2y}{dx^2} + 9y = 0, \quad y(0) = 1, \quad y(\pi/6) = 3.$

(*iii*) $\dfrac{d^2y}{dt^2} + 4\dfrac{dy}{dt} + 7y = 0, \quad y(0) = 0, \quad \dfrac{dy}{dt} = 3$ when $t = 0$.

(*iv*) $\dfrac{d^2x}{dt^2} - 4\dfrac{dx}{dt} + 4x = 0, \quad x(0) = 1, \quad x(1) = 3e^2.$

Solution

(*i*) Auxiliary equation is $\lambda^2 + \lambda - 20 = 0$ or $(\lambda + 5)(\lambda - 4) = 0$. So $\lambda = 4, -5$.
General solution is $y = Ae^{-5x} + Be^{4x}$.
When $x = 0, y = 1$ so $A + B = 1$.
Now $y' = \dfrac{dy}{dx} = -5Ae^{-5x} + 4Be^{4x}$. When $x = 0, y' = 1$ so $-5A + 4B = 1$.
Hence $B = 2/3, A = 1/3$.
Particular solution is $y = \dfrac{1}{3}e^{-5x} + \dfrac{2}{3}e^{4x}$.

(*ii*) Auxiliary equation is $\lambda^2 + 9 = 0$, so $\lambda = \pm 3i$.
General solution is $y = E\cos 3x + F\sin 3x$.
When $x = 0, y = 1$ so $E = 1$.
When $x = \frac{\pi}{6}, y = 3$ so $F = 3$.
Particular solution is $y = \cos 3x + 3\sin 3x$.

(*iii*) Auxiliary equation is $\lambda^2 + 4\lambda + 7 = 0$. So $\lambda = \dfrac{-4 \pm \sqrt{16 - 28}}{2} = -2 \pm \sqrt{3}i$.
General solution is $y = e^{-2t}(E\cos\sqrt{3}t + F\sin\sqrt{3}t)$.
When $t = 0, y = 0$ so $E = 0$.
When $t = 0, \dfrac{dy}{dt} = 3$. And for $E = 0$, $\dfrac{dy}{dt} = -2e^{-2t}F\sin\sqrt{3}t + \sqrt{3}e^{-2t}F\cos\sqrt{3}t$.
At $t = 0$, we have $3 = \sqrt{3}F$ so $F = \sqrt{3}$.
The particular solution is thus $y = \sqrt{3}e^{-2t}\sin\sqrt{3}t$.

(*iv*) Auxiliary equation is $\lambda^2 - 4\lambda + 4 = 0$ or $(\lambda - 2)^2 = 0$
Since $\lambda = 2$ is a repeated root, the general solution is $x = Ae^{2t} + Bte^{2t}$.
When $t = 0, \quad x = 1$ so $A = 1$.
When $t = 1, \quad x = 3e^2$ so $3e^2 = e^2 + Be^2$ or $B = 2$.
The particular solution is $x = e^{2t} + 2te^{2t}$.

Answers to Preparatory Questions

1. (*i*) The auxiliary equation $\lambda^2 + 2\lambda - 8 = 0$. Roots $\lambda = 2$ or $\lambda = -4$.

(*ii*) The auxiliary equation $\lambda^2 + 2\lambda - 4 = 0$. Roots $\lambda = -1 - \sqrt{5}$.

(*iii*) The auxiliary equation $\lambda^2 - 9 = 0$. Roots $\lambda = 3$ or $\lambda = -3$.

(*iv*) The auxiliary equation $\lambda^2 - 2\lambda + 5 = 0$. Roots $\lambda = 1 \pm 2i$.

(*v*) The auxiliary equation $\lambda^2 + 2\lambda + 1 = 0$. This has two equal roots $\lambda = -1$.