The University of Sydney
Math1003 Integral Calculus and Modelling

Semester 2 **Exercises and Solutions for Week 13** 2009

Assumed Knowledge

Objectives

(12a) To be able to rewrite two coupled first-order differential equations as a single second-order differential equation.

(12b) To be able to sketch the solutions of second-order differential equations with constant coefficients.

Preparatory Questions

1. Two species, struggling to compete against each other in the same environment, have populations at time t of $x(t)$ and $y(t)$, satisfying the equations

$$x'(t) = 3x(t) - 4y(t)\,, \qquad y'(t) = -2x(t) + y(t)\,.$$

Find the second-order differential equation satisfied by $x(t)$.

Practice Questions

2. Find $x(t)$ and $y(t)$ in Preparatory Question 1.

Solution For $x'' - 4x' - 5x = 0$ the auxiliary equation is $m^2 - 4m - 5 = 0$ with roots $m = 5$ or $m = -1$.
So $x = Ae^{5t} + Be^{-t}$. Hence $x' = 5Ae^{5t} - Be^{-t}$, and from $x' = 3x - 4y$ we obtain

$$\begin{aligned} y &= \tfrac{1}{4}(3x - x') \\ &= \tfrac{1}{4}[3(Ae^{5t} + Be^{-t}) - (5Ae^{5t} - Be^{-t})] \\ &= -\tfrac{1}{2}Ae^{5t} + Be^{-t}\,. \end{aligned}$$

3. Two species are in a predator-prey relationship. Let the predator species number Y, and the prey species number X individuals. Historically the numbers of these species have been constant at $X = 3000$ and $Y = 1500$. After a severe environmental disturbance the populations cease to be constant and start to change with time.

For some time variable t let $x(t)$ and $y(t)$ be the difference between the historically constant population numbers and the new, changing population numbers $X(t)$ and $Y(t)$. Then $x(t) = X(t) - 3000$ and $y(t) = Y(t) - 1500$ are the sizes of the perturbations from the historically steady states.
Suppose the perturbations satisfy:

$$\begin{aligned} x'(t) &= 3x(t) - 2y(t) \\ y'(t) &= 4x(t) - y(t). \end{aligned}$$

(*i*) Show that $x''(t) - 2x'(t) + 5x(t) = 0$.

(*ii*) Find $x(t)$ if $x(0) = 100$ and $x'(0) = 100$. (Take $t = 0$ to be the time at which monitoring of the population sizes begins.)

(*iii*) Hence find $y(t)$.

(*iv*) Sketch $x(t)$ and $y(t)$ and then $X(t)$ and $Y(t)$ as a function of t.
What does the model predict will happen to the original populations?

Solution

(*i*)

$$\begin{aligned} x' &= 3x - 2y \\ x'' &= 3x' - 2y' \\ &= 3x' - 2(4x - y) \\ &= 3x' - 8x + 2y \\ &= 3x' - 8x + (3x - x') \\ &= 2x' - 5x\,. \end{aligned}$$

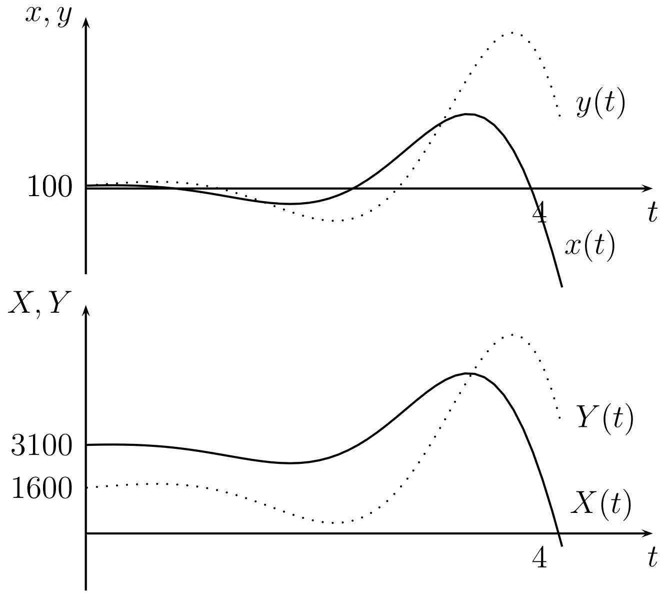
That is, $x'' - 2x' + 5x = 0$.

(*ii*) The auxiliary equation is $m^2 - 2m + 5 = 0$, with roots $m = 1 + 2i$ and $m = 1 - 2i$.
Therefore $x = e^t\,(A\cos 2t + B\sin 2t)$.
If $x = 100$ when $t = 0$, then $A = 100$ and $x = e^t(100\cos 2t + B\sin 2t)$.
Hence, $x' = e^t(-200\sin 2t + 2B\cos 2t) + e^t(100\cos 2t + B\sin 2t)$. When $t = 0$, $x' = 2B + 100 = 100$, and so $B = 0$. Therefore $x = 100e^t\cos 2t$.

(*iii*)

$$\begin{aligned} y(t) &= \tfrac{1}{2}(3x - x') \\ &= \tfrac{1}{2}(300e^t\cos 2t - 100e^t\cos 2t + 200e^t\sin 2t) \\ &= 100e^t(\cos 2t + \sin 2t)\,. \end{aligned}$$

(*iv*)



The populations will oscillate with increasingly large swings. Eventually the swings will get so big that $X(t) = 3000 + x(t)$ will be zero. Once this happens the model, and hence the solutions, will no longer be valid. The prey will then become extinct. The predator will then presumably also become extinct unless it has an alternative source of food.

More Questions

4. Find the general solution of the following system of equations:

$$\frac{dx}{dt} = x + y, \qquad \frac{dy}{dt} = 3x - y\,.$$

Solution We can rearrange this system to obtain one second-order differential equation. Differentiate the first equation with respect to t to obtain $\dfrac{d^2x}{dt^2} = \dfrac{dx}{dt} + \dfrac{dy}{dt}$
Substituting for $\dfrac{dy}{dt}$ from the second equation then gives $\dfrac{d^2x}{dt^2} = \dfrac{dx}{dt} + 3x - y$.
Finally, we can use the first equation of the system to replace y by $\dfrac{dx}{dt} - x$:

$$\frac{d^2x}{dt^2} = \frac{dx}{dt} + 3x - \left(\frac{dx}{dt} - x\right)\,.$$

Collecting terms on the left-hand-side then gives the second-order linear differential equation $\dfrac{d^2x}{dt^2} - 4x = 0$.
This has the auxiliary equation $m^2 - 4 = 0$, so $m = \pm 2$, and so we obtain the solution $x = Ae^{2t} + Be^{-2t}$.
From first equation $y = \dfrac{dx}{dt} - x = 2Ae^{2t} - 2Be^{-2t} - Ae^{2t} - Be^{-2t} = Ae^{2t} - 3Be^{-2t}$.
So the general solution of the system is

$$x = Ae^{2t} + Be^{-2t}, \qquad y = Ae^{2t} - 3Be^{-2t}$$

where A and B are arbitrary constants.

5. Find the general solution of the pair of differential equations

$$\frac{dx}{dt} = 5x - 3y\,, \qquad \frac{dy}{dt} = 2y\,,$$

by first solving the second equation and then substituting into the first. (There are two equations, so you should have *two* arbitrary constants of integration at the end.) Find the particular solution satisfying the initial conditions $x = 1$, $y = 2$ when $t = 0$.

Solution The second equation is separable and has the general solution $\ln|y| = 2t + C$, or $y = Ae^{2t}$.
Substitute this result into the first equation, and rearrange to obtain the equation $dx/dt - 5x = -3Ae^{2t}$. This is a first-order linear equation, and has the general solution $x = Ae^{2t} + Be^{5t}$.
If $y = 2$ when $t = 0$, we have $A = 2$. If $x = 1$ when $t = 0$, we have $1 = A + B$, and so $B = -1$.
The required particular solution is therefore $x = 2e^{2t} - e^{5t}\,, \quad y = 2e^{2t}$.

6. (*i*) For each of the following systems of differential equations for $x(t)$ and $y(t)$, find an equivalent second-order differential equation. (You do not need to find any solutions.)

(a) $x' = y + 4x, \qquad y' = 6x.$

(b) $x' = y, \qquad y' = x + 5y.$

(*ii*) Eliminate $y(t)$ from the following system to obtain a nonlinear second-order equation for $x(t)$. (You do not need to find any solutions.)

$$x' = y, \qquad y' = x - xy + 3t$$

Solution

(*i*) (a) Differentiating the first equation gives $x'' = y' + 4x'$. We can then replace y' by $6x$ to obtain $x'' = 6x + 4x'$. Thus we obtain the equivalent second-order equation $x'' - 4x' - 6x = 0$. Note that elimination of $x(t)$ gives the same equation for $y(t)$.

(b) Differentiating the second equation gives $y'' = x' + 5y'$. We can then replace x' by y to obtain $y'' = y + 5y'$. Thus we obtain the equivalent second-order equation $y'' - 5y' - y = 0$. Note that elimination of $y(t)$ gives the same equation for $x(t)$.

(*ii*) Differentiating the first equation gives $x'' = y'$. The second equation tells us $y' = x - xy + 3t$, so we obtain $x'' = x - xy + 3t$. We can then replace y by x' to get $x'' = x - xx' + 3t$. This then gives the nonlinear second-order differential equation $x'' + xx' - x = 3t$.

7. Find whether each of the following first-order equations is separable, linear or neither. If the equation is separable or linear find its general solution.

(*i*) $\dfrac{dy}{dx} = \dfrac{x^2 - y}{x}$ (*ii*) $\dfrac{dy}{dx} = \dfrac{y - 1}{x(1 + x)}$

(*iii*) $\dfrac{dy}{dx} = \dfrac{\sin x - y}{x - \cos y}$ (*iv*) $\dfrac{dy}{dx} = \dfrac{y^2 - 1}{2(1 + x)y}$

Solution

(*i*) The equation is not separable, however we can rearrange the equation as

$$\frac{dy}{dx} + \frac{y}{x} = x,$$

which we now recognise as first-order linear.
Multiplying by the integrating factor

$$e^{\int (1/x)dx} = e^{\ln x} = x.$$

we obtain

$$x\frac{dy}{dx} + y = x^2 \quad \text{i.e.} \quad \frac{d}{dx}(xy) = x^2.$$

Integrating both sides with respect to x, we get

$$xy = \frac{x^3}{3} + C.$$

Thus we obtain the general solution $y = \dfrac{x^2}{3} + \dfrac{C}{x}$.

(*ii*) The equation is linear (for y a function of x)

$$\frac{dy}{dx} - \frac{y}{x(1+x)} = \frac{-1}{x(1+x)},$$

and separable

$$\frac{dy}{dx} = \frac{y-1}{x(1+x)} = (y-1)\left(\frac{1}{x(1+x)}\right).$$

Separating and integrating,

$$\int \frac{dy}{y-1} = \int \frac{dx}{x(1+x)} = \int \left(\frac{1}{x} - \frac{1}{1+x}\right) dx \quad \text{(by partial fractions)}.$$

So

$$\ln|y-1| = \ln|x| - \ln|1+x| + C = \ln\left|\frac{x}{1+x}\right| + C.$$

Therefore,

$$y - 1 = \frac{Ax}{1+x}, \text{ where } A = \pm e^C.$$

So the general solution is $y = 1 + \dfrac{Ax}{1+x}$.

(*iii*) This equation is neither separable nor linear.

(*iv*) The equation is linear (for x a function of y),

$$\frac{dx}{dy} - \frac{2yx}{y^2-1} = \frac{2y}{y^2-1},$$

and separable

$$\frac{dy}{dx} = \frac{y^2-1}{2(1+x)y} = \left(\frac{y^2-1}{2y}\right)\left(\frac{1}{1+x}\right).$$

Separating and integrating:

$$\int \frac{2y}{y^2-1} dy = \int \frac{dx}{1+x}.$$

This gives $\ln|y^2-1| = \ln|1+x| + C$, and so $y^2 - 1 = A(1+x)$, where $A = \pm e^C$. Thus we obtain the general solution $y^2 = 1 + A(1+x)$.

Answers to Preparatory Questions

1. Differentiating $x' = 3x - 4y$ gives $x'' = 3x' - 4y'$. But $y' = -2x + y$, so

$$\begin{aligned} x'' &= 3x' - 4(-2x+y) \\ &= 3x' + 8x - 4y \\ &= 3x' + 8x + (x' - 3x) \quad (\text{since } -4y = x' - 3x) \\ &= 4x' + 5x\,. \end{aligned}$$

That is, $x'' - 4x' - 5x = 0$.