MATH 1003

Extended Answer Section

Answer these questions in the answer book(s) provided.
Ask for extra books if you need them.

1. (a) (*i*) Compute the upper and lower Riemann sum estimates for the definite integral $\int_1^3 \frac{1}{x(6-x)}\,dx$ using two equally spaced subintervals. (You may assume that the integrand is a decreasing function for $1 \le x \le 3$.) [2 marks]

As the integrand is decreasing on $[1, 3]$

$$U_2 = [f(1) + f(2)]\,\Delta x$$
$$= f(1) + f(2) \qquad (\Delta x = 1)$$
$$= \frac{1}{5} + \frac{1}{8} = \frac{13}{40}$$

and $L_2 = [f(2) + f(3)]\,\Delta x$
$$= \frac{1}{8} + \frac{1}{9} = \frac{17}{72}$$

(*ii*) Use partial fractions to find the integral $\int_1^3 \frac{1}{x(6-x)}\,dx$ exactly.
Combine this answer with results from the previous part (*i*) to show that

$$17/12 < \ln(5) < 39/20.$$

[3 marks]

$$\frac{1}{x(6-x)} = \frac{A}{x} + \frac{B}{6-x}$$

$\Rightarrow A(6-x) + Bx = 1$ (*) & then, equating coefficients

$$\left.\begin{aligned} 6A &= 1 \\ -A + B &= 0 \end{aligned}\right\} \quad \text{so } A = B = 1/6$$

and $\frac{1}{x(6-x)} = \frac{1}{6}\left(\frac{1}{x} + \frac{1}{6-x}\right)$

[Alternatively, from (*), put $x=0 \Rightarrow A = 1/6$
put $x = 6 \Rightarrow B = 1/6$]

Using the partial fraction representation in the integral:

$$\int_1^3 \frac{1}{x(6-x)}\,dx = \frac{1}{6}\int_1^3 \left(\frac{1}{x} + \frac{1}{6-x}\right)dx = \Big[\ln x - \ln(6-x)\Big]_1^3$$

$$= \frac{1}{6}\left[\ln 3 - \cancel{\ln 1} - \ln 3 + \ln 5\right]$$

i.e. $I = \int_1^3 \frac{1}{x(6-x)}\,dx = \frac{\ln 5}{6}$

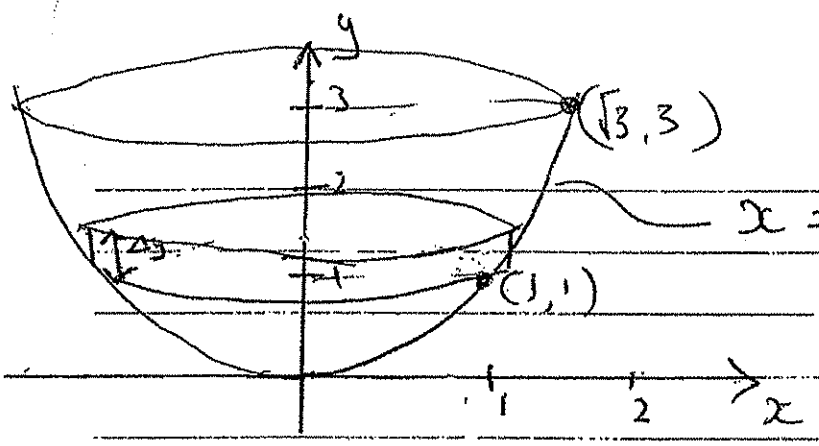
Finally as $L_2 < I < U_2$ (Strict inequalities since integrand is decreasing; accept $\leq$ with no penalty).

so $\frac{17}{72} < \frac{\ln 5}{6} < \frac{13}{40}$

i.e. $\frac{17}{12} < \ln 5 < \frac{78}{40} = \frac{39}{20}$

(b) Consider the region in the xy-plane bounded by the y-axis, the lines $y = 1$ and $y = 3$, and the graph of $y = x^2$. This region is rotated around the y-axis. Use the *disk method* to calculate the volume of this solid of revolution.

[3 marks]



$x = \sqrt{y} = f(y)$

[no diagram or derivation of the formula required]

$$\Delta V = \pi [f(y)]^2 \Delta y$$

$$= \pi y \Delta y$$

so $V = \pi \int_1^3 y \, dy$

then $V = \pi \left[\frac{y^2}{2} \right]_1^3 = \pi \left(\frac{9}{2} - \frac{1}{2} \right)$

$= 4\pi$

So the volume $V = 4\pi$

(c) (*i*) Use integration by parts to find the indefinite integral $\int \ln(x)\,dx$.

[2 marks]

We have $\int u v' dx = uv - \int v u' dx$
(integration by parts formula)

Here we choose:

$u(x) = \ln(x) \quad v(x) = x, \quad v'(x) = 1$

so $\int \ln x \times 1 \, dx = \ln x \times x$

$- \int x \frac{d \ln x}{dx} dx$

or $\int \ln x \, dx = x \ln x - \int \frac{x}{x} dx$

$= x \ln x - x + C$

(*ii*) Hence or otherwise find the definite integral $\int_0^1 \ln(1+x)\,dx$. [2 marks]

$I = \int_0^1 \ln(1+x)\,dx$

let $u = 1+x$, $du = dx$

$x=0 \Rightarrow u=1$, $x=1 \Rightarrow u=2$

so $I = \int_1^2 \ln u\,du$

$= \left[u \ln u - u\right]_1^2$ (from part (i))

$= 2\ln 2 - \cancel{\ln 1} - 2 + 1$

$= 2\ln 2 - 1$

so $\int_0^1 \ln(1+x)\,dx = 2\ln 2 - 1$

MATH 1003

2. (a) Determine the general solution $x(t)$ of the differential equation

$$t\frac{dx}{dt} = -4x + \frac{\sin(2t)}{t^3}.$$

[4 marks]

- STANDARD FORM: $\frac{dx}{dt} + p(t)\cdot x = q(t)$

$\Rightarrow \boxed{\frac{dx}{dt} + \frac{4}{t}\cdot x = \frac{\sin(2t)}{t^4}} \Rightarrow p(t) = \frac{4}{t},\ q(t) = \frac{\sin(2t)}{t^4}$

- I.F. $r(t) = e^{\int p(t)dt} = e^{4\int \frac{dt}{t}} = e^{4\ln t} = \boxed{t^4}$

- $\frac{d}{dt}(r(t)\cdot x) = r(t)\cdot q(t) \Rightarrow \boxed{r(t)\cdot x = \int r(t)q(t)\,dt}$

$\Rightarrow t^4\cdot x = \int \sin(2t)\,dt = -\frac{1}{2}\cos(2t) + C$

$\Rightarrow \boxed{x(t) = t^{-4}\left(C - \frac{1}{2}\cos(2t)\right)}$

(b) Determine the general solution of the differential equation

$$\frac{d^2y}{dx^2} + y = 0,$$

and find the particular solution which satisfies the boundary conditions

$$y(0) = 1 \text{ and } y(\pi/2) = \pi.$$

[3 marks]

AE: $m^2 + 1 = 0$

ROOTS: $\boxed{m = \pm i} = \alpha \pm i\beta \Rightarrow \alpha = 0, \beta = 1$

GENERAL SOLUTION: $\boxed{\begin{aligned} y(x) &= e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x) \\ &= C_1 \cos x + C_2 \sin x \end{aligned}}$

PARTICULAR SOLUTION: $y(0) = C_1 = 1$

$y(\pi/2) = C_2 = \pi$

$\Rightarrow \boxed{y(x) = \cos x + \pi \sin x}$

ALTERNATIVE

GENERAL SOLUTION: $\boxed{y(x) = C\cos(x - \varphi)}$

$\Rightarrow \left.\begin{aligned} y(0) &= C\cos\varphi = 1 \\ y(\tfrac{\pi}{2}) &= C\cos(\tfrac{\pi}{2} - \varphi) = C\sin\varphi = \pi \end{aligned}\right\}$

$\Rightarrow C^2 = 1 + \pi^2 \Rightarrow C = \pm\sqrt{1+\pi^2} \Rightarrow \varphi = \cos^{-1}\left(\frac{1}{\sqrt{1+\pi^2}}\right)$

$\Rightarrow y(x) = \sqrt{1+\pi^2}\cos\left(x - \cos^{-1}\left(\frac{1}{\sqrt{1+\pi^2}}\right)\right) = \boxed{\cos x + \pi \sin x}$

(c) Determine the general solution of the following system of equations

$$\frac{dx}{dt} = 5x - 4y,$$
$$\frac{dy}{dt} = -8x + y$$

and find the particular solution satisfying the initial conditions

$$x = 2 \text{ and } y = 4 \text{ at } t = 0.$$

[5 marks]

$$\left.\begin{aligned} x' &= ax + by \\ y' &= cx + dy \end{aligned}\right\} \Rightarrow x'' - (a+d)x' + (ad-bc)x = 0$$ (*SEE REMARK BELOW)

$$\Rightarrow \boxed{x'' - 6x' - 27x = 0}$$

- AE: $m^2 - 6m - 27 = 0$
- ROOTS: $m_{1/2} = \frac{1}{2}(6 \pm \sqrt{36+108}) = \boxed{9, -3}$

$\Rightarrow$ GENERAL SOLUTION: $\boxed{x(t) = C_1 e^{9t} + C_2 e^{-3t}}$

- $x' = 5x - 4y \Rightarrow y = \frac{1}{4}(5x - x')$

$$\Rightarrow y(t) = \frac{1}{4}(5C_1 e^{9t} + 5C_2 e^{-3t} - 9C_1 e^{9t} + 3C_2 e^{-3t})$$
$$= -C_1 e^{9t} + 2C_2 e^{-3t}$$

$$\Rightarrow x(t) = C_1 e^{9t} + C_2 e^{-3t}; \boxed{y(t) = -C_1 e^{9t} + 2C_2 e^{-3t}}$$

IC: $\left.\begin{aligned} x(0) &= C_1 + C_2 = 2 \\ y(0) &= -C_1 + 2C_2 = 4 \end{aligned}\right\} \Rightarrow C_1 = 0, C_2 = 2$

$$\Rightarrow \boxed{x(t) = 2e^{-3t}; y(t) = 4e^{-3t}}$$

THE UNIVERSITY OF SYDNEY

MATH1003, Integral Calculus and Modelling (Normal)
Solution to Question 3 on 2011 Exam

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3. (a) [4 marks.] Use integration to find the area of the region enclosed by the circle $x^2 + y^2 = 4$ and the vertical line $x = 1$, where the region corresponds to $x \geq 1$.

(b) (*i*). [3 marks.] Evaluate exactly the definite integral,

$$J_1 = \int_0^1 (1 - x^2) \cos\left(\frac{\pi x}{2}\right) dx.$$

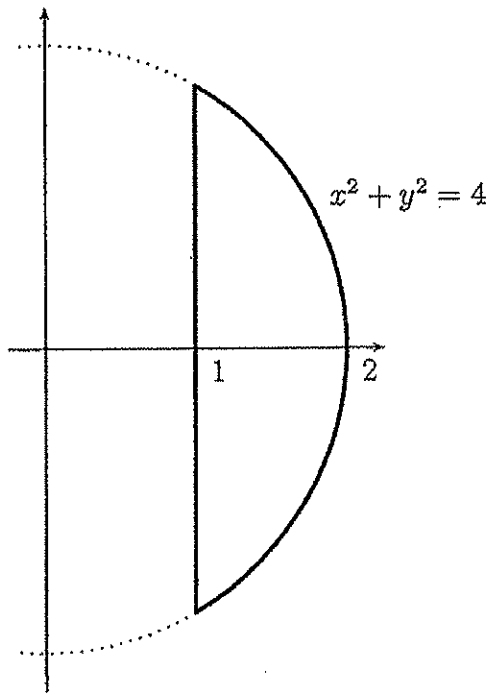
(*ii*). [5 marks.] [Harder.] Obtain a reduction formula for

$$J_n = \int_0^1 (1 - x^2)^n \cos\left(\frac{\pi x}{2}\right) dx, \qquad n = 0, 1, 2, \ldots.$$

Express your answer as a formula for J_n in terms of J_{n-1} and J_{n-2}, valid for $n \geq 2$.

Solution.

(a) We require the area of the following circular segment:



The area is given by the integral,

$$A = 2\int_1^2 \sqrt{4-x^2}\,dx.$$

Use the substitution $x = 2\sin\theta$, $dx = 2\cos\theta\,d\theta$. As with any integration by substitution, the limits of integration can be changed along with the substitution or entered at the end. The former method is the best here. So $x = 1$ corresponds to $\theta = \pi/6$ and $x = 2$ corresponds to $\theta = \pi/2$. The integral becomes

$$\begin{aligned}
A &= 2\int_{\pi/6}^{\pi/2} (2\cos\theta)(2\cos\theta)\,d\theta \\
&= \int_{\pi/6}^{\pi/2} 8\cos^2\theta\,d\theta \\
&= \int_{\pi/6}^{\pi/2} (4 + 4\cos 2\theta)\,d\theta \\
&= \Big[4\theta + 2\sin 2\theta\Big]_{\pi/6}^{\pi/2} \\
&= 2\pi + 2\sin\pi - \frac{2\pi}{3} - 2\sin\frac{\pi}{3} \\
&= \boxed{\frac{4\pi}{3} - \sqrt{3}.}
\end{aligned}$$

An alternative method is to calculate the indefinite integral,

$$\begin{aligned}
\int 2\sqrt{4-x^2}\,dx &= \int 8\cos^2\theta\,d\theta \\
&= 4\theta + 2\sin 2\theta + C \qquad \text{(as above)} \\
&= 4\theta + 4\sin\theta\cos\theta + C \\
&= 4\sin^{-1}\frac{x}{2} + x\sqrt{4-x^2} + C,
\end{aligned}$$

where "$+C$" is optional, given that it always cancels out of definite integrals. Entering the limits of integration gives

$$\begin{aligned}
A &= \left[4\sin^{-1}\frac{x}{2} + x\sqrt{4-x^2}\right]_1^2 \\
&= 4\sin^{-1}1 + 0 - 4\sin^{-1}\tfrac{1}{2} - \sqrt{3} \\
&= \boxed{\frac{4\pi}{3} - \sqrt{3}.}
\end{aligned}$$

(b) (i). In the integration by parts formula,

$$\int UV'\,dx = UV - \int U'V\,dx,$$

take $U = 1 - x^2$ and $V' = \cos(\pi x/2)$. These are the only reasonable choices. Then $U' = -2x$ and $V = (2/\pi)\sin(\pi x/2)$. The integral for J_1 becomes

$$\begin{aligned} J_1 &= \int_0^1 (1 - x^2)\cos\frac{\pi x}{2}\,dx \\ &= \left[\frac{2}{\pi}(1 - x^2)\sin\frac{\pi x}{2}\right]_0^1 - \int_0^1 \frac{2}{\pi}(-2x)\sin\frac{\pi x}{2}\,dx \\ &= \frac{4}{\pi}\int_0^1 x\sin\frac{\pi x}{2}\,dx. \end{aligned}$$

A second integration by parts is needed. Let $U = x$ and $V' = \sin(\pi x/2)$. Again, these choices are unique except for possible constant multipliers. Then $U' = 1$ and $V = -(2/\pi)\cos(\pi x/2)$. We get

$$\begin{aligned} J_1 &= \frac{4}{\pi}\int_0^1 x\sin\frac{\pi x}{2}\,dx \\ &= \frac{4}{\pi}\left[-\frac{2x}{\pi}\cos\frac{\pi x}{2}\right]_0^1 - \frac{4}{\pi}\int_0^1 \left(-\frac{2}{\pi}\right)\cos\frac{\pi x}{2}\,dx \\ &= \frac{8}{\pi^2}\int_0^1 \cos\frac{\pi x}{2}\,dx \\ &= \frac{8}{\pi^2}\left[\frac{2}{\pi}\sin\frac{\pi x}{2}\right]_0^1 \\ &= \boxed{\frac{16}{\pi^3}.} \end{aligned}$$

(*ii*). The reduction formula for J_n can be derived by integrating by parts twice. In the integration by parts formula,

$$\int UV'\,dx = UV - \int U'V\,dx,$$

take $U = (1-x^2)^n$ and $V' = \cos(\pi x/2)$. Then $U' = -2nx(1-x^2)^{n-1}$ and $V = (2/\pi)\sin(\pi x/2)$. The integral for J_n becomes

$$\begin{aligned} J_n &= \int_0^1 (1-x^2)^n \cos\frac{\pi x}{2}\,dx \\ &= \left[\frac{2}{\pi}(1-x^2)^n \sin\frac{\pi x}{2}\right]_0^1 - \int_0^1 \frac{2}{\pi}(-2nx)(1-x^2)^{n-1}\sin\frac{\pi x}{2}\,dx \\ &= \frac{4n}{\pi}\int_0^1 x(1-x^2)^{n-1}\sin\frac{\pi x}{2}\,dx, \qquad (*) \end{aligned}$$

provided $n > 0$ (which implies $n \geq 1$ since n is an integer). The condition on n is required because we assumed that $(1-x^2)^n$ vanishes when $x = 1$.

In the next integration by parts, let

$$U = x(1-x^2)^{n-1}, \qquad V' = \sin\frac{\pi x}{2}.$$

Then, with the help of the product rule, we get

$$U' = (1-x^2)^{n-1} - 2(n-1)x^2(1-x^2)^{n-2}, \qquad V = -\frac{2}{\pi}\cos\frac{\pi x}{2}.$$

Given that we want to express J_n in terms of J_{n-1} and J_{n-2} for $n \geq 2$, it will save time if we write $x^2 = 1-(1-x^2)$ and thereby rearrange U' as

$$\begin{aligned} U' &= (1-x^2)^{n-1} - 2(n-1)(1-x^2)^{n-2} + 2(n-1)(1-x^2)^{n-1} \\ &= (2n-1)(1-x^2)^{n-1} - 2(n-1)(1-x^2)^{n-2}. \end{aligned}$$

Then the second integration by parts becomes

$$
\begin{aligned}
J_n &= \frac{4n}{\pi}\int_0^1 x(1-x^2)^{n-1}\sin\frac{\pi x}{2}\,dx \\
&= \frac{4n}{\pi}\left[-\frac{2}{\pi}x(1-x^2)^{n-1}\cos\frac{\pi x}{2}\right]_0^1 \\
&\quad -\frac{4n}{\pi}\int_0^1\left(-\frac{2}{\pi}\right)\{(2n-1)(1-x^2)^{n-1} \\
&\quad -2(n-1)(1-x^2)^{n-2}\}\cos\frac{\pi x}{2}\,dx \\
&= \frac{8n}{\pi^2}\left\{(2n-1)\int_0^1(1-x^2)^{n-1}\cos\frac{\pi x}{2}\,dx\right. \\
&\quad \left.-2(n-1)\int_0^1(1-x^2)^{n-2}\cos\frac{\pi x}{2}\,dx\right\} \\
&= \boxed{\frac{8n}{\pi^2}\{(2n-1)J_{n-1}-2(n-1)J_{n-2}\}.}
\end{aligned}
$$

This is the required reduction formula for J_n. All the steps are valid for $n \geq 2$. (It also gives the correct result $J_1 = (8/\pi^2)J_0 = 16/\pi^3$ when $n = 1$ if we ignore the term involving J_{-1}, which has zero coefficient.)

It is not likely that many students will do the above rearrangement of U'. If we use the original form of U' instead, the second integration by parts reads

$$
\begin{aligned}
J_n &= \frac{4n}{\pi}\int_0^1 x(1-x^2)^{n-1}\sin\frac{\pi x}{2}\,dx \\
&= \frac{4n}{\pi}\left[-\frac{2}{\pi}x(1-x^2)^{n-1}\cos\frac{\pi x}{2}\right]_0^1 \\
&\quad -\frac{4n}{\pi}\int_0^1\left(-\frac{2}{\pi}\right)\{(1-x^2)^{n-1}-2(n-1)x^2(1-x^2)^{n-2}\}\cos\frac{\pi x}{2}\,dx \\
&= \frac{8n}{\pi^2}\left\{\int_0^1(1-x^2)^{n-1}\cos\frac{\pi x}{2}\,dx\right. \\
&\quad \left.-2(n-1)\int_0^1 x^2(1-x^2)^{n-2}\cos\frac{\pi x}{2}\,dx\right\} \\
&= \frac{8n}{\pi^2}\left\{J_{n-1}-2(n-1)\int_0^1 x^2(1-x^2)^{n-2}\cos\frac{\pi x}{2}\,dx\right\}.
\end{aligned}
$$

To finish off, one must recognise that $x^2 = 1 - (1 - x^2)$, which implies that

$$x^2(1 - x^2)^{n-2} = (1 - x^2)^{n-2} - (1 - x^2)^{n-1}.$$

This, in turn, implies that

$$\int_0^1 x^2(1 - x^2)^{n-2} \cos \frac{\pi x}{2}\, dx = J_{n-2} - J_{n-1}.$$

Hence the last result for J_n simplifies to

$$\begin{aligned} J_n &= \frac{8n}{\pi^2}\big\{J_{n-1} - 2(n-1)(J_{n-2} - J_{n-1})\big\} \\ &= \boxed{\frac{8n}{\pi^2}\big\{(2n-1)J_{n-1} - 2(n-1)J_{n-2}\big\},} \end{aligned}$$

in agreement with the previous method.