The University of Sydney
Math1003 Integral Calculus and Modelling

Semester 2 **Exercises and Solutions for Week 5** 2014

Assumed Knowledge: Sketching curves of simple functions. Integrals of simple functions such as x^n (including $1/x$), $\sin x$, $\cos x$, e^x.

Objectives:

(4a) To understand and be able to use integration by parts to evaluate definite integrals.

(4b) To understand that an indefinite integral is a function.

(4c) To understand that differentiation and (indefinite) integration are inverse processes when applied to functions.

(4d) To be able to sketch a function given its derivative.

(4e) To be able to derive a reduction formula for an integral.

Preparatory questions:

1. Find the indefinite integrals

(*i*) $\displaystyle\int \tan x\, dx$. Hint: Write $\tan x = \sin x/\cos x$ and choose a suitable substitution.

(*ii*) $\displaystyle\int x^2 e^x\, dx$.

Practice Questions:

2. Evaluate the following integrals by using integration by parts.

(*i*) $\displaystyle\int_0^{1/2} xe^{2x}\, dx$. (*ii*) $\displaystyle\int_0^{\pi/4} \theta \sin 4\theta\, d\theta$. (*iii*) $\displaystyle\int_1^2 t^2 \ln t\, dt$.

Solution

(*i*) Choose $u = x$ and $dv = e^{2x}dx$. Then $du = dx$ and $v = \frac{1}{2}e^{2x}dx$. So,

$$\int_0^{1/2} xe^{2x}\, dx = \left[x\frac{1}{2}e^{2x}\right]_0^{1/2} - \int_0^{1/2} \frac{1}{2}e^{2x}dx = \left[x\frac{1}{2}e^{2x}\right]_0^{1/2} - \left[\frac{1}{4}e^{2x}\right]_0^{1/2} = \frac{1}{4}\,.$$

(*ii*) Choose $u = \theta$ and $dv = \sin 4\theta \, d\theta$. Then $du = d\theta$ and $v = -\frac{1}{4}\cos 4\theta$. So,

$$\int_0^{\pi/4} \theta \sin 4\theta \, d\theta = \left[\theta\left(-\frac{1}{4}\cos 4\theta\right)\right]_0^{\pi/4} - \int_0^{\pi/4}\left(-\frac{1}{4}\cos 4\theta\right) d\theta$$
$$= \left[\theta\left(-\frac{1}{4}\cos 4\theta\right)\right]_0^{\pi/4} + \left[\frac{1}{16}\sin 4\theta\right]_0^{\pi/4} = \frac{\pi}{16}.$$

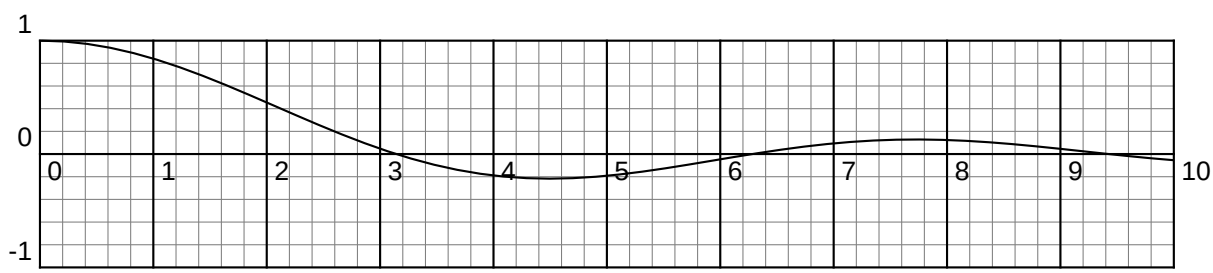
(*iii*)

$$\int_1^2 t^2 \ln t \, dt = \left[\frac{1}{3}t^3 \ln t\right]_1^2 - \int_1^2 \frac{1}{3}t^3 \frac{1}{t} dt = \left[\frac{1}{3}t^3 \ln t\right]_1^2 - \left[\frac{t^3}{9}\right]_1^2 = \frac{8}{3}\ln 2 - \frac{7}{9}.$$

3. Define $\mathrm{Si}(x)$ as $\mathrm{Si}(x) = \displaystyle\int_0^x f(t)\, dt$, where $f(t) = \begin{cases} \dfrac{\sin t}{t} & t \neq 0 \\ 1 & t = 0. \end{cases}$

This function is called the *sine-integral*, and is useful in optics.

This is the graph of $f(t)$.



(*i*) What is $\mathrm{Si}\,'(x)$?.

(*ii*) What is the value of $\mathrm{Si}(0)$?

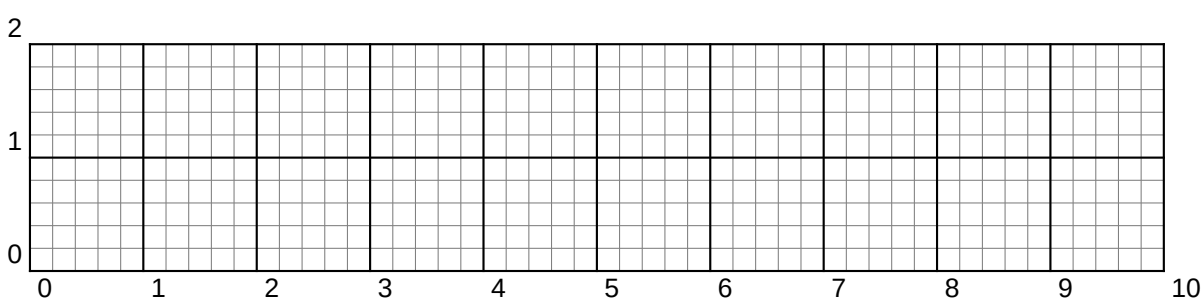
(*iii*) For $0 \leq x \leq 3\pi$, use the graph of $f(t)$ to determine the values of x for which $\mathrm{Si}(x)$ is increasing, and the values of x for which it is decreasing.

(*iv*) For which values of x between 0 and 3π does $\mathrm{Si}(x)$ have stationary points?

(*v*) Use the graph of $f(t)$ to estimate $\mathrm{Si}(\pi)$, $\mathrm{Si}(2\pi)$ and $\mathrm{Si}(3\pi)$.

(*vi*) Use the graph of $f(t)$ to determine values of x at which Si has points of inflection.

(*vii*) Sketch the graph of Si for $0 \leq x \leq 3\pi$.



Solution

(*i*) By the Fundamental Theorem of Calculus Part I, $\mathrm{Si}'(x) = \dfrac{\sin x}{x}$ for $x \neq 0$ and $\mathrm{Si}'(x) = 1$ when $x = 0$.

(*ii*) $\mathrm{Si}(0) = \int_0^0 f(t)\,dt = 0.$

(*iii*) From (a), we have $\mathrm{Si}'(x) = (\sin x)/x$. Now $\mathrm{Si}(x)$ is increasing when $\mathrm{Si}'(x) > 0$, and decreasing when $\mathrm{Si}'(x) < 0$, so we need to determine the values of x for which $(\sin x)/x$ is positive, and those for which it is negative. Looking at the graph provided, or noting that for $x > 0$ we have $(\sin x)/x > 0$ precisely when $\sin x > 0$ and $(\sin x)/x < 0$ precisely when $\sin x < 0$, we see that

$$\begin{aligned}(\sin x)/x > 0, &\quad \text{for } 0 < x < \pi \text{ and } 2\pi < x < 3\pi\\ (\sin x)/x < 0, &\quad \text{for } \pi < x < 2\pi.\end{aligned}$$

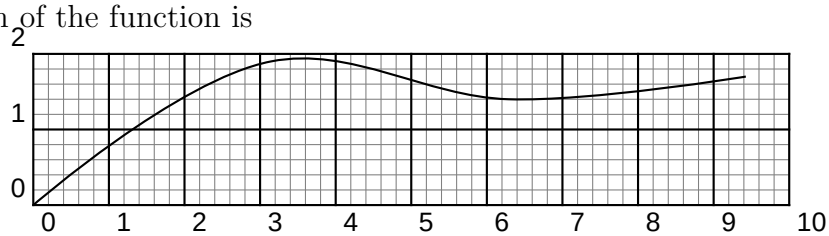
So $\mathrm{Si}(x)$ is increasing for $0 < x < \pi$ and $2\pi < x < 3\pi$, and decreasing for $\pi < x < 2\pi$.

(*iv*) $\mathrm{Si}(x)$ has stationary points when $\mathrm{Si}'(x) = 0$. Looking at the graph again, or noting that $(\sin x)/x = 0$ precisely when $\sin x = 0$ (except at $x = 0$), we find the stationary points of $\mathrm{Si}(x)$ to be at $x = \pi, 2\pi$, and 3π.

(*v*) Add up the squares under the graph of $(\sin x)/x$ to estimate $\mathrm{Si}(x)$, remembering that the area of each square is $0.2 \times 0.2 = 0.04$ and that the area under the x axis is given a negative sign. This gives $\mathrm{Si}(\pi) \doteq 1.9$, $\mathrm{Si}(2\pi) \doteq 1.4$, $\mathrm{Si}(3\pi) \doteq 1.7$. Note that these are approximate, and you may have obtained slightly different values.

(*vi*) Since $(\sin x)/x$ is the derivative of Si, the points of inflection of Si occur when $(\sin x)/x$ has stationary points. That is, when $x = 0$, 4.5, 7.7.

(*vii*) The sketch of the function is



4. Establish the following reduction formula. [Hint: Write the integrand as $u(x)v'(x)$ where $u = \cos^{n-1} x$.]

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx\,.$$

Use this formula to find $\int \cos^2 x\,dx$ and $\int \cos^4 x\,dx$.

Solution Putting $u = \cos^{n-1} x$ and $v' = \cos x$ so that $v = \sin x$, we can integrate by parts:

$$\begin{aligned}I_n = \int \cos^n x\,dx &= \int \cos^{n-1} x \frac{d}{dx}(\sin x)\,dx\\ &= \cos^{n-1} x \sin x - \int \sin x \frac{d}{dx}(\cos^{n-1} x)\,dx\\ &= \cos^{n-1} x \sin x - \int \sin x (n-1)\cos^{n-2} x(-\sin x)\,dx\\ &= \cos^{n-1} x \sin x + (n-1)\int \sin^2 x \cos^{n-2} x\,dx\\ &= \cos^{n-1} x \sin x + (n-1)\int (1-\cos^2 x)\cos^{n-2} x\,dx\\ &= \cos^{n-1} x \sin x + (n-1)I_{n-2} - (n-1)I_n\,.\end{aligned}$$

Rearranging,

$$nI_n = \cos^{n-1} x \sin x + (n-1)I_{n-2} \quad \text{or} \quad I_n = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n} I_{n-2}\,.$$

Now $I_0 = \int \cos^0 x\, dx = \int dx = x + C$, so

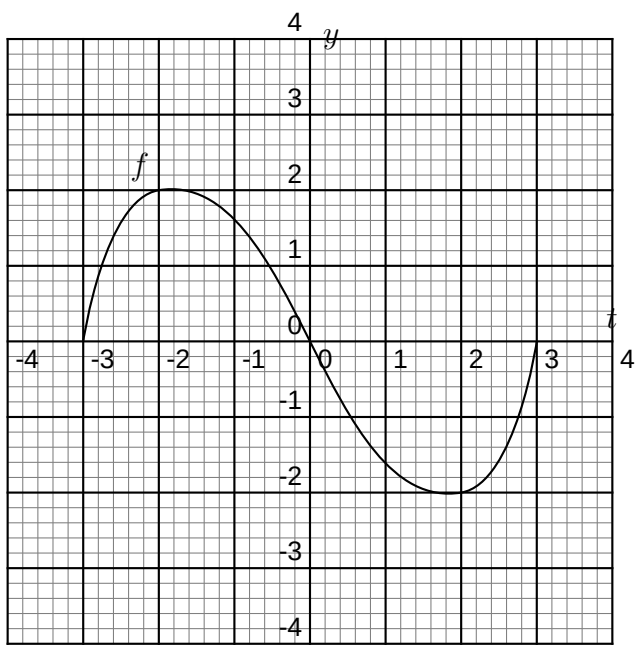
$$I_2 = \frac{1}{2}\cos x \sin x + \frac{1}{2} I_0 = \frac{1}{2}\cos x \sin x + \frac{1}{2}x + C_1$$

and

$$I_4 = \frac{1}{4}\cos^3 x \sin x + \frac{3}{4} I_2 = \frac{1}{4}\cos^3 x \sin x + \frac{3}{8}\cos x \sin x + \frac{3}{8}x + C_2\,.$$

More Exercises

5. Let $g(x) = \int_{-3}^{x} f(t)\, dt$ where f is the *odd* function whose graph is shown.



(*i*) Evaluate $g(-3)$ and $g(3)$.

(*ii*) Estimate $g(-2)$, $g(-1)$ and $g(0)$.

(*iii*) On what interval is g increasing?

(*iv*) Where does g have a maximum value?

(*v*) Sketch a rough graph of g.

Solution

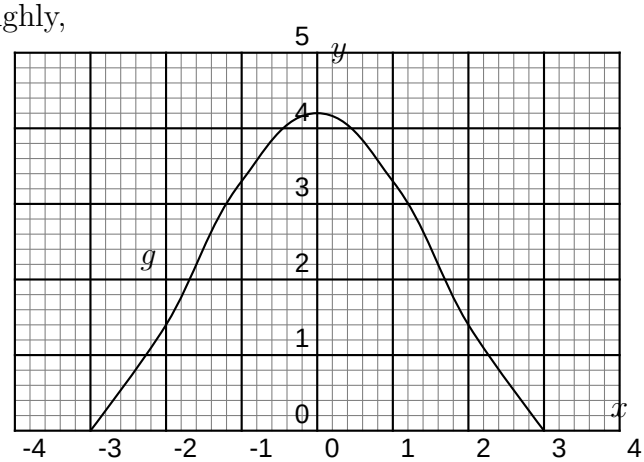
(*i*) $g(-3) = 0$, $g(3) = 0$.

(*ii*) $g(-2) \approx 1.4$, $g(-1) \approx 3.3$, $g(0) \approx 4.2$.

(*iii*) $(-3, 0)$.

(*iv*) At $t = 0$.

(v) The sketch is roughly,



6. Evaluate the following integrals by using integration by parts.

(i) $\displaystyle\int_0^1 (2x+3)e^x\,dx.$ $\qquad$ (ii) $\displaystyle\int_0^\pi \theta^2\cos 3\theta\,d\theta.$ $\qquad$ (iii) $\displaystyle\int_{-\pi/4}^{\pi/4} t\sin t\cos t\,dt.$

Hint: First use an identity in (iii).

Solution

(i)

$$\int_0^1 (2x+3)e^x\,dx = [(2x+3)e^x]_0^1 - \int_0^1 2e^x dx = [(2x+3)e^x]_0^1 - [2e^x]_0^1 = 3e-1\,.$$

(ii) First choose $u=\theta^2$ and $dv=\cos 3\theta\,d\theta$. So $du = 2\theta\,d\theta$ and $v=\frac{1}{3}\sin 3\theta$.

$$\begin{aligned}\int_0^\pi \theta^2\cos 3\theta\,d\theta &= \left[\theta^2\frac{1}{3}\sin 3\theta\right]_0^\pi - \int_0^\pi 2\theta\frac{1}{3}\sin 3\theta\,d\theta\\ &= \left[\frac{1}{3}\theta^2\sin 3\theta\right]_0^\pi - \frac{2}{3}\int_0^\pi \theta\sin 3\theta\,d\theta\,.\end{aligned}$$

Now we need to integrate by parts again, this time choosing $u=\theta$ and $dv = \sin 3\theta\,d\theta$.

$$\begin{aligned}\int_0^\pi \theta\sin 3\theta\,d\theta &= \left[\theta\left(-\frac{1}{3}\cos 3\theta\right)\right]_0^\pi - \int_0^\pi \left(-\frac{1}{3}\cos 3\theta\right)\,d\theta\\ &= \left[-\frac{1}{3}\theta\cos 3\theta\right]_0^\pi + \left[\frac{1}{9}\sin 3\theta\right]_0^\pi\,.\end{aligned}$$

So

$$\int_0^\pi \theta^2\cos 3\theta\,d\theta = \left[\frac{1}{3}\theta^2\sin 3\theta + \frac{2}{9}\theta\cos 3\theta - \frac{2}{27}\sin 3\theta\right]_0^\pi = -\frac{2\pi}{9}\,.$$

(iii) First use the identity $\sin 2t = 2\sin t\cos t$ and then integration by parts.

$$\begin{aligned}\int_{-\pi/4}^{\pi/4} t\sin t\cos t\,dt &= \frac{1}{2}\int_{-\pi/4}^{\pi/4} t\sin 2t\,dt\\ &= \frac{1}{2}\left(\left[t\left(-\frac{1}{2}\cos 2t)\right)\right]_{-\pi/4}^{\pi/4} - \int_{-\pi/4}^{\pi/4}\left(-\frac{1}{2}\cos 2t\right)\,dt\right)\\ &= \left[-\frac{1}{4}t\cos 2t + \frac{1}{8}\sin 2t\right]_{-\pi/4}^{\pi/4} = \frac{1}{4}\,.\end{aligned}$$

7. Let $I_n = \int x^n e^x dx$. Use integration by parts to establish the reduction formula

$$I_n = x^n e^x - nI_{n-1}\,.$$

Hence find $\int x^3 e^x dx$.

Solution Choose $u = x^n$ and $dv/dx = e^x$, i.e. $v = e^x$. Then

$$I_n = \int x^n \frac{d}{dx} e^x\, dx = x^n e^x - \int n x^{n-1} e^x\, dx = x^n e^x - nI_{n-1}\,.$$

To find I_3, first find $I_0 = \int x^0 e^x\, dx = \int e^x\, dx = e^x + C$. Then

$$I_1 = xe^x - I_0 = xe^x - e^x + C_1\,,$$

and

$$I_2 = x^2 e^x - 2I_1 = x^2 e^x - 2xe^x + 2e^x + C_2\,,$$

and

$$I_3 = x^3 e^x - 3I_2 = x^3 e^x - 3x^2 e^x + 6xe^x - 6e^x + C_3\,.$$

8. (i) Let $I_n = \int x(\ln x)^n dx$. Use integration by parts to establish the reduction formula

$$I_n = \frac{1}{2}x^2(\ln x)^n - \frac{n}{2}I_{n-1}\,.$$

(ii) Starting with $I_0 = \int x\, dx = \frac{1}{2}x^2 + C$, use the reduction formula from part (i) to find I_2.

Solution

(i) Since we cannot easily integrate $(\ln x)^n$, we choose $u = (\ln x)^n$ and $v'(x) = x$, i.e. $v(x) = x^2/2$. Then

$$\begin{aligned}
\int x(\ln x)^n\, dx &= \int (\ln x)^n \frac{d}{dx}\left(\frac{x^2}{2}\right) dx \\
&= \frac{1}{2}x^2(\ln x)^n - \int \frac{x^2}{2}\frac{d}{dx}(\ln x)^n\, dx \\
&= \frac{1}{2}x^2(\ln x)^n - \int \frac{x^2}{2} n(\ln x)^{n-1}\frac{1}{x}\, dx \\
&= \frac{1}{2}x^2(\ln x)^n - \frac{n}{2}\int x(\ln x)^{n-1}\, dx = \frac{1}{2}x^2(\ln x)^n - \frac{n}{2}I_{n-1}\,.
\end{aligned}$$

(ii) Starting with $I_0 = \frac{1}{2}x^2 + C$,

$$I_1 = \frac{1}{2}x^2(\ln x) - \frac{1}{2}I_0 = \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C_1\,,$$

where C_1 is a constant. Applying the formula again,

$$I_2 = \frac{1}{2}x^2(\ln x)^2 - \frac{2}{2}I_1 = \frac{1}{2}x^2(\ln x)^2 - \frac{1}{2}x^2 \ln x + \frac{1}{4}x^2 + C_2\,.$$

Answers to Preparatory Questions

1. (i) $-\ln|\cos x| + C$. (ii) $(x^2 - 2x + 2)e^x + C$.