The University of Sydney
Math1003 Integral Calculus and Modelling

Semester 2 **Exercises and Solutions for Week 9** 2014

Assumed Knowledge: Integration techniques.

Objectives:

(8a) To be able to write a rational expression as a sum of partial fractions.

(8b) To be able to solve a separable equation using partial fractions.

Preparatory questions:

1. Use partial fractions to express the following as a sum of terms:

(i) $\dfrac{1}{(x+1)(x-2)}$ (ii) $\dfrac{7}{2x^2+5x-3}$

(iii) $\dfrac{1}{(x+a)(x+b)}$ (iv) $\dfrac{x}{x-5}$

Practice Questions:

2. Evaluate the integrals:

(i) $\displaystyle\int \frac{1}{(x+1)(x-2)}dx$ (ii) $\displaystyle\int \frac{7}{2x^2+5x-3}dx$

(iii) $\displaystyle\int \frac{1}{(x+a)(x+b)}dx$ (iv) $\displaystyle\int \frac{x}{x-5}dx$

Solution

(i) From Question 1 part (i),

$$\begin{aligned}\int \frac{1}{(x+1)(x-2)}dx &= \int \left(\frac{\frac{1}{3}}{x-2} - \frac{\frac{1}{3}}{x+1}\right) dx \\ &= \frac{1}{3}\ln|x-2| - \frac{1}{3}\ln|x+1| + C \\ &= \frac{1}{3}\ln\left|\frac{x-2}{x+1}\right| + C\,.\end{aligned}$$

(ii) From Question 1 part (ii),

$$\begin{aligned}\int \frac{7}{2x^2+5x-3}dx &= \int \left(\frac{2}{2x-1} - \frac{1}{x+3}\right) dx \\ &= \ln|2x-1| - \ln|x+3| + C \\ &= \ln\left|\frac{2x-1}{x+3}\right| + C\,.\end{aligned}$$

(*iii*) From Question 1 part (*iii*),

$$\begin{aligned}\int \frac{1}{(x+a)(x+b)}dx &= \int \frac{1}{b-a}\left(\frac{1}{x+a} - \frac{1}{x+b}\right)dx \\ &= \frac{1}{b-a}\left(\ln|x+a| - \ln|x+b|\right) + C \\ &= \frac{1}{b-a}\ln\left|\frac{x+a}{x+b}\right| + C\,.\end{aligned}$$

(*iv*) From Question 1 part (*iv*),

$$\begin{aligned}\int \frac{x}{x-5}dx &= \int \left(1 + \frac{5}{x-5}\right)dx \\ &= x + 5\ln|x-5| + C\,.\end{aligned}$$

3. (Suitable for group work and discussion.) A trout farmer believes that, if she doesn't remove any fish from a certain tank, the population of fish is modelled by the differential equation

$$\frac{dP}{dt} = 2P - 0.01P^2,$$

where P is the number of fish after t years.

(*i*) Suppose the farmer initially stocks the tank with 50 fish.

(a) What is the maximum number of fish the tank can support?

(b) Find P for which the population is increasing most rapidly.

(c) Find an explicit formula for P in terms of t.

(d) How long will it be before there are 199 fish in the tank?

(e) Sketch a graph of $y = P(t)$.

(*ii*) Now suppose that the farmer initially stocks the tank with 250 fish. Find the solution $P(t)$, and sketch its graph.

(*iii*) If the trout farmer were to remove 75 fish from the tank each year, then the differential equation modelling population size would be

$$\frac{dP}{dt} = 2P - 0.01P^2 - 75.$$

(a) Find a solution to this differential equation, supposing that the farmer initially stocks the tank with 60 fish.

(b) According to this model, what is the size of the fish population in the long-term?

(c) What should the farmer expect to happen if she initially stocks the tank with 50 fish?

Solution

(*i*) (a) Note that $P = 200$ is an equilibrium solution, since $\frac{dP}{dt} = 0$ if $P = 200$. If $P > 200$, $\frac{dP}{dt} < 0$, and P decreases to 200. Hence the maximum number of fish that the tank can support is 200.

(b) We have to maximise $\dfrac{dP}{dt} = 0.01P(200-P)$. This is a quadratic function, whose graph is a concave down parabola passing through the P axis at 0 and 200. For such a parabola, the maximum occurs half way between the two P intercepts. So the population increases most rapidly when $P = \frac{1}{2}(0+200) = 100$.

(c) Separating the variables, the equation can be written as

$$\int \frac{dP}{P(200-P)} = \int 0.01\,dt\,.$$

Partial fractions gives

$$\frac{1}{P(200-P)} = \frac{1}{200P} + \frac{1}{200(200-P)},$$

and so the above equation becomes

$$\frac{1}{200}\int \left(\frac{1}{P} + \frac{1}{200-P}\right) dP = \int 0.01\,dt.$$

Therefore

$$\frac{1}{200}\ln\left|\frac{P}{200-P}\right| = 0.01t + C, \qquad \text{or} \qquad \frac{P}{200-P} = Ae^{2t} \quad (A = \pm e^{200C}).$$

Since $P(0) = 50$, $A = \frac{50}{150} = \frac{1}{3}$. Hence $\dfrac{P}{200-P} = \dfrac{e^{2t}}{3}$, and

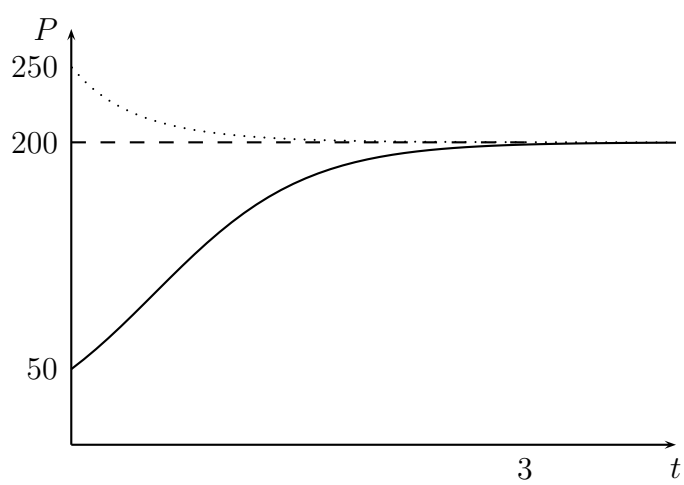
$$P(t) = \frac{200}{1+3e^{-2t}}\,.$$

(d) Use the equation $\dfrac{P}{200-P} = \dfrac{e^{2t}}{3}$ to find t at which $P = 199$.

With $P = 199$, we have $\dfrac{199}{200-199} = \dfrac{e^{2t}}{3}$.

So $e^{2t} = 3 \times 199$, and $t = \frac{1}{2}\ln(597) \cong 3.196$. That is, it takes just over 3 years.

(e)



(*ii*) We use the general solution $\dfrac{P}{200-P} = Ae^{2t}$, as in (*i*)(c), and the initial condition $P(0) = 250$. This gives $A = -5$, and

$$P = \frac{200}{1 - \frac{1}{5}e^{-2t}}.$$

The graph is the dotted curve in the above diagram.

(*iii*) (a) Separating variables, the equation can be written as:

$$\int \frac{dP}{(P-150)(P-50)} = \int -0.01\,dt.$$

Using partial fractions, we have

$$\frac{1}{(P-150)(P-50)} = \frac{1}{100(P-150)} - \frac{1}{100(P-50)}.$$

So the above equation becomes

$$\frac{1}{100}\int \left(\frac{1}{P-150} - \frac{1}{P-50}\right) dP = \int -0.01\,dt.$$

Hence

$$\frac{1}{100}\ln\left|\frac{P-150}{P-50}\right| = -0.01t + C, \quad \text{or} \quad \frac{P-150}{P-50} = Ae^{-t}.$$

Since $P = 60$ when $t = 0$, we have

$$A = \frac{60-150}{60-50} = -9, \text{ and so } \frac{P-150}{P-50} = -9e^{-t}.$$

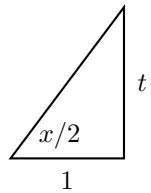
On rearranging this we obtain

$$P = \frac{150(1+3e^{-t})}{1+9e^{-t}}.$$

(b) For the long term behaviour, we let $t \to \infty$. Since $e^{-t} \to 0$ as $t \to \infty$, we have $P \to \frac{150(1+0)}{1+0} = 150$.

(c) Note that $\dfrac{dP}{dt} = -0.01(P-150)(P-50) = 0$ when $P = 50$. So if the farmer initially stocks 50 fish, the population will be at an equilibrium and she should expect the number to remain at 50.

4. (*i*) Use the geometry of the following right-angled triangle in which $\tan(x/2) = t$



to show that

(a) $\sin\dfrac{x}{2} = \dfrac{t}{\sqrt{1+t^2}}$ (b) $\cos\dfrac{x}{2} = \dfrac{1}{\sqrt{1+t^2}}$.

(*ii*) Hence show that

(a) $\cos x = \dfrac{1-t^2}{1+t^2}$ (b) $\sin x = \dfrac{2t}{1+t^2}$.

(*iii*) If $t = \tan\dfrac{x}{2}$, show that $\dfrac{dx}{dt} = \dfrac{2}{1+t^2}$.

(*iv*) Use the substitution $t = \tan\dfrac{x}{2}$ to evaluate

$$\int \frac{5}{4\sin x + 3\cos x}dx.$$

Solution

(*i*) The hypoteneuse is $\sqrt{1+t^2}$ so the results follow from the definitions of tan, cos and sin in terms of the ratios of sides of a right-angled triangle.

(*ii*) (a) $\cos x = \cos^2\dfrac{x}{2} - \sin^2\dfrac{x}{2} = \dfrac{1}{1+t^2} - \dfrac{t^2}{1+t^2} = \dfrac{1-t^2}{1+t^2}$.

(b) $\sin x = 2\sin\dfrac{x}{2}\cos\dfrac{x}{2} = 2\left(\dfrac{t}{\sqrt{1+t^2}}\right)\left(\dfrac{1}{\sqrt{1+t^2}}\right) = \dfrac{2t}{1+t^2}$.

(*iii*) $\dfrac{dx}{dt} = \dfrac{d}{dt}\left(2\tan^{-1}t\right) = 2\left(\dfrac{1}{1+t^2}\right) = \dfrac{2}{1+t^2}$.

(*iv*) If $t = \tan\frac{x}{2}$, then $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$ and $dx = \frac{2}{1+t^2}dt$. So

$$\begin{aligned}\int \frac{5}{4\sin x + 3\cos x}dx &= \int \frac{5\frac{2}{1+t^2}}{4\frac{2t}{1+t^2} + 3\frac{1-t^2}{1+t^2}}dt \\ &= \int \frac{10}{8t+3-3t^2}dt = -\int \frac{10}{(3t+1)(t-3)}dt\,.\end{aligned}$$

Use partial fractions to express

$$\frac{10}{(3t+1)(t-3)} = \frac{A}{3t+1} + \frac{B}{t-3} \quad \text{or} \quad 10 = A(t-3) + B(3t+1).$$

Putting $t = 3$ we obtain $10 = 10B$ so $B = 1$ and putting $t = -\frac{1}{3}$ we obtain $10 = A(-\frac{10}{3})$ so $A = -3$. Thus

$$\begin{aligned}\int \frac{5}{4\sin x + 3\cos x}dx &= -\int \left(\frac{-3}{3t+1} + \frac{1}{t-3}\right)dt \\ &= \ln|3t+1| - \ln|t-3| + C \\ &= \ln\left|\frac{3t+1}{t-3}\right| + C = \ln\left|\frac{3\tan(x/2)+1}{\tan(x/2)-3}\right| + C\,.\end{aligned}$$

More Exercises

5. (*i*) An algal bloom will occur in water polluted by excess nutrients when the number of algae cells x increases dramatically. The rate of increase of the number of algae cells in a sample of such water during a bloom is proportional to the number of cells present at any instant. Write down a differential equation that models this phenomenon.

(*ii*) If, however, the water is low in oxygen then the rate of increase is proportional not to the total number of algae cells x but instead, at any time t ($t \geq 0$), to $e^{-at}x$, where a is a positive constant. [That is, the rate of increase is proportional to a fraction of the total number of cells, and that fraction decreases exponentially with time.]

(a) Write down a differential equation that expresses these statements.

(b) Solve it to obtain the number of cells in a sample as a function of time, given the initial number of algae cells.

(c) Is there a limit to the number of cells as $t \to \infty$?

Solution

(*i*) We have that $dx/dt \propto x$, i.e. $dx/dt = kx$, where k is the constant of proportionality.

(*ii*) (a) Now we have $dx/dt \propto e^{-at}x$, i.e. $dx/dt = ke^{-at}x$, where k is the constant of proportionality.

(b) Separate the equation:

$$\int \frac{1}{x} dx = \int ke^{-at} dt$$

so

$$\ln x = -\frac{k}{a}e^{-at} + C\,,$$

and so

$$x = \exp\left(-\frac{k}{a}e^{-at} + C\right) = A\exp\left(-\frac{k}{a}e^{-at}\right)\,.$$

Putting $x = x_0$ at $t = 0$ we find $x_0 = A\exp(-k/a)$.

So $A = x_0 \exp\left(\frac{k}{a}\right)$ and

$$x = x_0 \exp\left(\frac{k}{a}\right)\exp\left(\frac{-k}{a}e^{-at}\right) = x_0 \exp\left[\left(\frac{k}{a}\right)(1 - e^{-at})\right]\,.$$

(c) As $t \to \infty$, $e^{-at} \to 0$ and so x tends to a limiting value of $x_0 e^{k/a}$.

Answers to Preparatory Questions

1. (*i*) $\frac{1}{3}\left(\frac{1}{x-2} - \frac{1}{x+1}\right)$ (*ii*) $\frac{2}{2x-1} - \frac{1}{x+3}$

(*iii*) $\frac{1}{b-a}\left(\frac{1}{x+a} - \frac{1}{x+b}\right)$ (*iv*) $1 + \frac{5}{x-5}$