

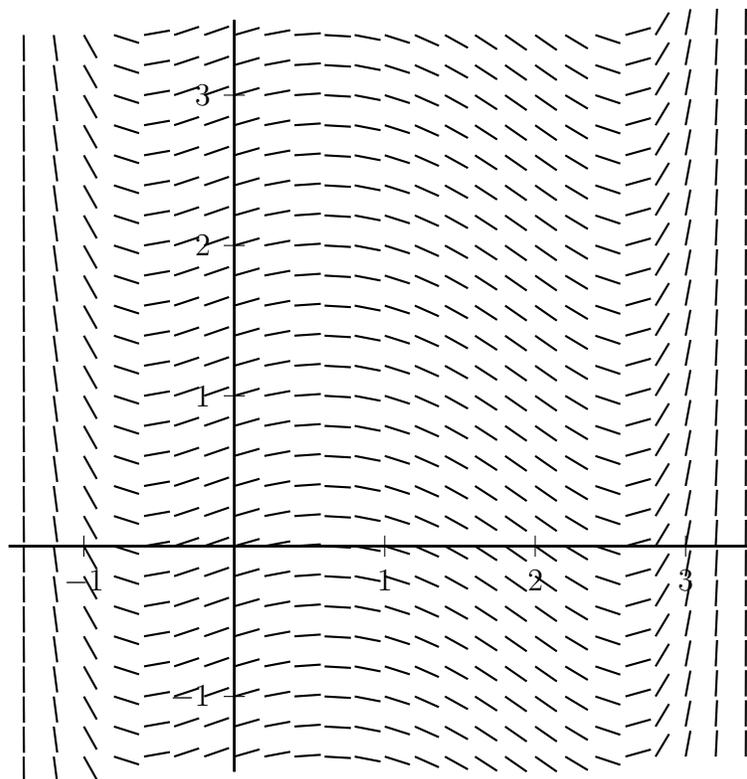
Assumed Knowledge: Proportionality and inverse proportionality. Integration techniques. Taylor series and binomial series.

Objectives:

- (6a) Given a verbal description of a simple model, to be able to express it as a mathematical equation.
- (6b) To be able to recognise an ordinary differential equation.
- (6c) To be able to sketch the solution curves for a first-order differential equation from its direction field.

Preparatory questions:

1. (i) The differential equation $\frac{dy}{dx} = f(x)$ has a direction field given by the diagram below.
On the direction field draw the graphs of two solutions of $dy/dx = f(x)$, where one solution $y = g(x)$ passes through the point $(0, 1)$ and the other solution $y = h(x)$ satisfies the equation $h(1) = 0$.
- (ii) Do the graphs of $y = g(x)$ and $y = h(x)$ intersect? If not, why not?



Practice Questions:

2. Construct a mathematical model of the following experiment (given below) by:

- (i) determining and naming suitable independent and dependent variables;
- (ii) constructing a differential equation;
- (iii) relating the other conditions to the variables you have introduced.

Heat tends to flow from hot bodies to cold bodies. Newton observed that the rate at which temperature rises or falls within a body is proportional to the temperature difference between the body and the surrounding air. In the experiment to test Newton's law, the surrounding air remained at a constant temperature and the body temperature was initially twice the temperature of the surrounding air.

3. (i) Find the general solution by antidifferentiation and sketch the solution curves of:

(a) $\frac{dy}{dx} = \cos 2x$ (b) $\frac{dy}{dx} = \cosh x$

(ii) Find the particular solution of

$$\frac{dy}{dx} = \frac{1}{1+x^2}, \quad y(1) = \pi/4.$$

4. (Suitable for group work and discussion.) Newton's law of gravitation states that the acceleration of an object at a distance r from the centre of an object of mass M is given by

$$\frac{d^2r}{dt^2} = -\frac{GM}{r^2},$$

where G is the universal gravitational constant.

(i) Use the identity

$$\frac{d^2r}{dt^2} = \frac{d}{dr} \left(\frac{1}{2}v^2 \right),$$

where $v = dr/dt$, to show by integration that

$$v^2 - u^2 = \frac{2GM}{r} - \frac{2GM}{R},$$

when $v = u$ at $r = R$.

(ii) Now write $r = R + s$ where s is the height of the object above the surface of the Earth, radius R and mass M . Use the binomial series to expand the factor $(1 + s/R)^{-1}$ to show that, close to the surface of the Earth,

$$v^2 \simeq u^2 - 2gs,$$

for some constant g .

Find the expression for g .

More Exercises

