

## Week 6 Exercises and Objectives

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MATH1003: Integral Calculus and Modelling

Semester 2, 2015

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Lecturers: B. Crossman, R. Marangell, M. Myerscough, S. Stephen

**Assumed Knowledge:** Proportionality and inverse proportionality. Integration techniques. Taylor series and binomial series.

### Objectives:

- (6a) Given a verbal description of a simple model, to be able to express it as a mathematical equation.
- (6b) To be able to recognise an ordinary differential equation.
- (6c) To be able to sketch the solution curves for a first-order differential equation from its direction field.

### Exercises:

1. Find the general solution by antidifferentiation and sketch the solution curves of:

(a)  $\frac{dy}{dx} = \cos 2x$

(b)  $\frac{dy}{dx} = \cosh x$

2. Construct a mathematical model of the following experiment (given below) by:

- (i) determining and naming suitable independent and dependent variables;
- (ii) constructing a differential equation;
- (iii) relating the other conditions to the variables you have introduced.

Heat tends to flow from hot bodies to cold bodies. Newton observed that the rate at which temperature rises or falls within a body is proportional to the temperature difference between the body and the surrounding air. In the experiment to test Newton's law, the surrounding air remained at a constant temperature and the body temperature was initially twice the temperature of the surrounding air.

3. (a) Show that  $y = A(e^x - x)$ , where  $A$  is an arbitrary constant, satisfies the differential equation

$$\frac{dy}{dx} = \frac{y(e^x - 1)}{e^x - x}.$$

- (b) Since the general solution of a first order-differential equation involves only one arbitrary constant, we see that the solution given in part (a) is the general solution of the differential equation. Find the particular solution satisfying the initial condition  $y(0) = -3$ .
- (c) Sketch the solution curves given in part (a) in the special cases  $A = 1$ ,  $A = 0$  and  $A = -1$ .

4. Find the general solution of the following differential equations, and in each case give also the particular solution satisfying the initial condition  $y(0) = 2$ .

(a)  $\frac{dy}{dx} = 20xe^{5x^2}$

(c)  $\frac{dy}{dx} = x \cos x$

(b)  $\frac{dy}{dx} = 6x^2 + \cos x$

(d)  $\frac{dy}{dx} = x^2e^x$

5. Newton's law of gravitation states that the acceleration of an object at a distance  $r$  from the centre of an object of mass  $M$  is given by

$$\frac{d^2r}{dt^2} = -\frac{GM}{r^2},$$

where  $G$  is the universal gravitational constant.

- (a) Use the identity

$$\frac{d^2r}{dt^2} = \frac{d}{dr} \left( \frac{1}{2}v^2 \right),$$

where  $v = dr/dt$ , to show by integration that

$$v^2 - u^2 = \frac{2GM}{r} - \frac{2GM}{R},$$

when  $v = u$  at  $r = R$ .

- (b) Now write  $r = R + s$  where  $s$  is the height of the object above the surface of the Earth, radius  $R$  and mass  $M$ . Use the binomial series to expand the factor  $(1 + s/R)^{-1}$  to show that, close to the surface of the Earth,

$$v^2 \approx u^2 - 2gs,$$

for some constant  $g$ .

Find the expression for  $g$ .

6. *Please note that this question is a little harder than the previous ones.*

A ball has the property that each time it strikes a hard, level surface with velocity  $v$ , it rebounds with velocity  $-kv$ , where  $0 < k < 1$ . Suppose that the ball is dropped from an initial height of  $H$  metres above the surface.

- (a) Use the formula  $s = ut - gt^2/2$ , where  $u$  is the initial velocity and  $g$  is the acceleration of gravity, to find the time taken for it to hit the ground.
- (b) Use the formula  $v = u - gt$  to find the velocity with which it hits the surface.
- (c) Find the velocity with which it rebounds.
- (d) Use the formula  $s = ut - gt^2/2$  again to find the time taken for the ball to fall back to the surface after the first bounce.
- (e) What is the velocity of the ball just before the second bounce and what is the rebound velocity after the second bounce? Find the time taken to fall back to the surface after the second bounce.
- (f) Write down the series for the time taken from the moment the ball is dropped until the fifth bounce.
- (g) How long is it before the ball comes to rest?