THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Week 10 Exercises and Objectives

MATH1003: Integral Calculus and Modelling — Semester 2, 2015

Web Page: www.sydney.edu.au/science/maths/u/UG/JM/MATH1003/
Lecturers: B. Crossman, R. Marangell, M. Myerscough, S. Stephen

Assumed Knowledge: Integration techniques.

Objectives:

(10a) To be able to solve differential equations that are separable, linear or both.

(10b) To be able to construct and solve equations describing flow and mixing problems.

Exercises:

1. Classify and solve the differential equation $(1+x^2)\dfrac{dy}{dx} = y - 2xy$.

2. In a prolific breed of rabbits, the birth and death rates are each proportional to the square of the population N. Let k_1, k_2 be the constants of proportionality for births and deaths respectively. Assume $k_1 > k_2$ and write $k = k_1 - k_2$. Then $dN/dt = kN^2$, where $k > 0$, and t is the time in months. Solve this differential equation to show that

$$N(t) = \frac{N_0}{1 - kN_0 t},$$

where N_0 is the initial population.
Suppose that $N_0 = 2$ and that there are 4 rabbits after 3 months. What does this model predict will happen after another 3 months?

3. Radiocarbon dating allows us to estimate the age of ancient objects. In living organisms, the ratio of radioactive carbon-14 to ordinary carbon-12 is constant. However when the organism dies, carbon-14 is no longer absorbed (from the atmosphere or via feeding for example), and so the amount present decreases through radioactive decay. By comparing the amount of carbon-14 present with the amount which would normally be present, we can determine the number of years since an organism died (or the age of objects such as clothing or paper, made from once-living material).
The half-life of carbon-14 is 5730 years. This means that the rate of decay is proportional to the amount $y(t)$ of carbon-14 present,

$$\frac{dy}{dt} = -ky \qquad (k > 0),$$

and that half of any given amount will disintegrate in 5730 years.

(a) Find the solution of this equation given that at time $t = 0$, the amount of carbon-14 present is y_0.

(b) Use the fact that half of this amount y_0 will disintegrate over 5730 years to find a numerical value for k.

(c) A piece of woollen clothing is found to have only 77% of the amount of carbon-14 normally found in wool. Estimate the age of this piece of clothing.

4. A lucerne crop on an experimental farm is grown on an unirrigated paddock. The rate of growth of the crop depends on the mass of the lucerne plants and the water content of the soil. The water content of the soil declines exponentially with time when there is no rain. Once the soil water content falls below a critical level W_c the crop stops growing and starts to die back.
The soil moisture content $M(t)$ is modelled as $M(t) = Ke^{-at}$.

(a) Sketch M as a function of t.

(b) Let s be the mass of the crop in kilograms per hectare and let t be the time in weeks. The following equation is proposed as a model for this crop's growth:

$$\frac{ds}{dt} = m(Ke^{-at} - W_c)s \qquad (m \text{ a constant}).$$

Explain why this equation might be a reasonable model for crop growth.

(c) Find the time t when the crop stops growing and starts to die back.

(d) Find the general solution to the differential equation in part (b).

An agricultural scientist has data on this crop in this paddock from the previous seasons. He knows that $m = 0.035$, $W_c = 1.3$ and $a = 0.41$.

(e) A crop is planted and starts to grow. Then heavy rain falls and increases the soil moisture content to 15 times W_c, the critical level. Let $t = 0$ immediately after the rain. Use the expression for soil moisture $M(t)$ to find K.

(f) How soon after the rain does the crop stop growing?

(g) When the rain falls the crop has grown to a mass of 400 kg per hectare. What is the maximum mass per hectare attained by the crop if there is no further rain? If the crop is cut 12 weeks after the rain what will be the yield per hectare?

5. A holding pond at a chemical treatment plant contains 2000 litres of water which has been contaminated with a particular impurity. The concentration of this impurity in the pond is initially 5 grams per litre. Water is drained from the pond for treatment at a rate of 10 litres per minute and water from another pond is added at the same rate. This input water also contains the impurity at a concentration of 0.02 grams per litre. A safe level of this impurity is considered to be 1 gram per litre. Formulate a differential equation for the rate of change of the mass of impurity in the holding pond. Assuming that the water in the holding pond is well-stirred, how long does it take for the impurity to reach safe levels?