Assumed Knowledge: Finding the roots of quadratic equations. Euler’s formula $e^{i\theta} = \cos \theta + i\sin \theta$.

Objectives:

(11a) To be able to write down the auxiliary (or characteristic) equation associated with a second-order differential equation with constant coefficients.

(11b) To be able to construct the solutions to such differential equations in terms of exponential and trigonometric functions.

Exercises:

1. Find the general solution of $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 4y = 0$.
   
   **Solution:** The auxiliary equation is $\lambda^2 + 2\lambda - 4 = 0$ so that $\lambda = -1 \pm \sqrt{5}$.
   
   Hence $y = Ae^{(-1+\sqrt{5})t} + Be^{(-1-\sqrt{5})t} = e^{-t}(Ae^{\sqrt{5}t} + Be^{-\sqrt{5}t})$.

2. Find the particular solution of $\frac{d^2y}{dt^2} - 9y = 0$ which satisfies the initial conditions $y = 3$ and $\frac{dy}{dt} = 3$ when $t = 0$.
   
   **Solution:** The auxiliary equation is $\lambda^2 - 9 = 0$ so that $\lambda = 3$ or $\lambda = -3$.
   
   The general solution is therefore $y = Ae^{3t} + Be^{-3t}$.
   
   Now, $\frac{dy}{dt} = 3Ae^{3t} - 3Be^{-3t}$, and so when $t = 0$, $y = A + B = 3$ and $\frac{dy}{dt} = 3A - 3B = 3$. Solving these equations gives $A = 2$ and $B = 1$, so the particular solution is $y = 2e^{3t} - e^{-3t}$.

3. Find the general solutions of these second-order homogeneous equations. In each case give your answer in terms of real functions.
   
   (a) $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 3y = 0$.
      
      **Solution:** Auxiliary equation is $2\lambda^2 - 7\lambda + 3 = 0$ or $(2\lambda - 1)(\lambda - 3) = 0$.
      
      So $\lambda = \frac{1}{2}, 3$.
      
      So the general solution is $y = Ae^{x/2} + Be^{3x}$.

   (b) $\frac{d^2y}{dx^2} + 3y = 0$.
      
      **Solution:** Auxiliary equation is $\lambda^2 + 3 = 0$. So $\lambda = \pm i\sqrt{3}$.
      
      So the general solution is $y = A \cos \sqrt{3}x + B \sin \sqrt{3}x$. 
(c) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$.

**Solution:** Auxiliary equation is $\lambda^2 - 2\lambda + 2 = 0$. So $\lambda = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$. Therefore the general solution is $y = e^x(A\cos x + B\sin x)$.

4. Find the general solution of each of these second-order linear homogeneous equations. Hence find the particular solution for the given conditions. In each case give your answer in terms of real functions.

(a) $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 20y = 0$, $y(0) = y'(0) = 1$.

**Solution:** Auxiliary equation is $\lambda^2 + \lambda - 20 = 0$ or $(\lambda + 5)(\lambda - 4) = 0$. So $\lambda = 4, -5$.
General solution is $y = Ae^{-5x} + Be^{4x}$.
When $x = 0, y = 1$ so $A + B = 1$.
Now $y' = \frac{dy}{dx} = -5Ae^{-5x} + 4Be^{4x}$. When $x = 0, y' = 1$ so $-5A + 4B = 1$.
Hence $B = \frac{2}{3}, A = \frac{1}{3}$.
Particular solution is $y = \frac{1}{3}e^{-5x} + \frac{2}{3}e^{4x}$.

(b) $\frac{d^2y}{dx^2} + 9y = 0$, $y(0) = 1$, $y(\pi/6) = 3$.

**Solution:** Auxiliary equation is $\lambda^2 + 9 = 0$, so $\lambda = \pm 3i$.
General solution is $y = E\cos 3x + F\sin 3x$.
When $x = 0, y = 1$ so $E = 1$.
When $x = \frac{\pi}{6}, y = 3$ so $F = 3$.
Particular solution is $y = \cos 3x + 3\sin 3x$.

(c) $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 7y = 0$, $y(0) = 0$, $\frac{dy}{dt} = 3$ when $t = 0$.

**Solution:** Auxiliary equation is $\lambda^2 + 4\lambda + 7 = 0$. So $\lambda = \frac{-4 \pm \sqrt{16-28}}{2} = -2 \pm \sqrt{3}i$.
General solution is $y = e^{-2t}(E\cos \sqrt{3}t + F\sin \sqrt{3}t)$.
When $t = 0, y = 0$ so $E = 0$.
When $t = 0, \frac{dy}{dt} = 3$ so calculate $\frac{dy}{dt} = -2e^{-2t}F\sin \sqrt{3}t + \sqrt{3}e^{-2t}F\cos \sqrt{3}t$, since $E = 0$.
At $t = 0$, we have $3 = \sqrt{3}F$ so $F = \sqrt{3}$. The particular solution is thus $y = \sqrt{3}e^{-2t}\sin \sqrt{3}t$.

(d) $\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 4x = 0$, $x(0) = 1$, $x(1) = 3e^2$.

**Solution:** Auxiliary equation is $\lambda^2 - 4\lambda + 4 = 0$ or $(\lambda - 2)^2 = 0$.
Since $\lambda = 2$ is a repeated root, the general solution is $x = Ae^{2t} + Bte^{2t}$.
When $t = 0$, $x = 1$ so $A = 1$.
When $t = 1$, $x = 3e^2$ so $3e^2 = e^2 + Be^2$ or $B = 2$.
The particular solution is $x = e^{2t} + 2te^{2t}$.