THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Solutions to Week 12 Exercises and Objectives

MATH1003: Integral Calculus and Modelling — Semester 2, 2015

Web Page: www.sydney.edu.au/science/maths/u/UG/JM/MATH1003/
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Assumed Knowledge: Solving simultaneous linear equations.

Objectives:

(12a) To be able to rewrite two coupled first-order differential equations as a single second-order differential equation.

(12b) To be able to sketch the solutions of second-order differential equations with constant coefficients.

Exercises:

1. Find the general solution of the following system of equations:

$$\frac{dx}{dt} = x + y, \qquad \frac{dy}{dt} = 3x - y\,.$$

Solution: We can rearrange this system to obtain one second-order differential equation. Differentiate the first equation with respect to t to obtain $\dfrac{d^2x}{dt^2} = \dfrac{dx}{dt} + \dfrac{dy}{dt}$.
Substituting for $\dfrac{dy}{dt}$ from the second equation then gives $\dfrac{d^2x}{dt^2} = \dfrac{dx}{dt} + 3x - y$.
Finally, we can use the first equation of the system to replace y by $\dfrac{dx}{dt} - x$:

$$\frac{d^2x}{dt^2} = \frac{dx}{dt} + 3x - \left(\frac{dx}{dt} - x\right)\,.$$

Collecting terms on the left-hand-side then gives the second-order linear differential equation $\dfrac{d^2x}{dt^2} - 4x = 0$.
This has the auxiliary equation $m^2 - 4 = 0$, so $m = \pm 2$, and so we obtain the solution $x = Ae^{2t} + Be^{-2t}$.
From the first equation, $y = \dfrac{dx}{dt} - x = 2Ae^{2t} - 2Be^{-2t} - Ae^{2t} - Be^{-2t} = Ae^{2t} - 3Be^{-2t}$.
So the general solution of the system is

$$x = Ae^{2t} + Be^{-2t}, \qquad y = Ae^{2t} - 3Be^{-2t}$$

where A and B are two arbitrary constants.

2. Find the general solution of the pair of differential equations

$$\frac{dx}{dt} = 5x - 3y\,, \qquad \frac{dy}{dt} = 2y\,,$$

by first solving the second equation and then substituting into the first. (There are two equations, so you should have *two* arbitrary constants of integration at the end.)
Find the particular solution satisfying the initial conditions $x = 1$, $y = 2$ when $t = 0$.

Solution: The second equation is separable and has the general solution $\ln|y| = 2t + C$, or $y = Ae^{2t}$.
Substitute this result into the first equation, and rearrange to obtain the equation $dx/dt - 5x = -3Ae^{2t}$. This is a first-order linear equation, and has the general solution $x = Ae^{2t} + Be^{5t}$.
If $y = 2$ when $t = 0$, we have $A = 2$. If $x = 1$ when $t = 0$, we have $1 = A + B$, and so $B = -1$.
The required particular solution is therefore $x = 2e^{2t} - e^{5t}\,, \quad y = 2e^{2t}$.

3. (a) For each of the following systems of differential equations for $x(t)$ and $y(t)$, find an equivalent second-order differential equation. (You do not need to find any solutions.)

(i) $x' = y + 4x, \qquad y' = 6x.$

Solution: Differentiating the first equation gives $x'' = y' + 4x'$. We can then replace y' by $6x$ to obtain $x'' = 6x + 4x'$. Thus we obtain the equivalent second-order equation $x'' - 4x' - 6x = 0$. Note that elimination of $x(t)$ gives the same equation for $y(t)$.

(ii) $x' = y, \qquad y' = x + 5y.$

Solution: Differentiating the second equation gives $y'' = x' + 5y'$. We can then replace x' by y to obtain $y'' = y + 5y'$. Thus we obtain the equivalent second-order equation $y'' - 5y' - y = 0$. Note that elimination of $y(t)$ gives the same equation for $x(t)$.

(b) Eliminate $y(t)$ from the following system to obtain a nonlinear second-order equation for $x(t)$. (You do not need to find any solutions.)

$$x' = y, \qquad y' = x - xy + 3t.$$

Solution: Differentiating the first equation gives $x'' = y'$. The second equation tells us $y' = x - xy + 3t$, so we obtain $x'' = x - xy + 3t$. We can then replace y by x' to get $x'' = x - xx' + 3t$. This then gives the nonlinear second-order differential equation $x'' + xx' - x = 3t$.