The University of Sydney
School of Mathematics and Statistics

Solutions to Week 13 Tutorial

MATH1003: Integral Calculus and Modelling — Semester 2, 2015

Web Page: www.sydney.edu.au/science/maths/u/UG/JM/MATH1003/
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This tutorial covers material from the lectures in Week 12.

1. Two species struggling to compete against each other in the same environment have populations at time t of $x(t)$ and $y(t)$, satisfying the equations

$$x'(t) = 3x(t) - 4y(t)\,, \qquad y'(t) = -2x(t) + y(t)\,.$$

Find the second-order differential equation satisfied by $x(t)$.
Hence find $x(t)$ and $y(t)$.

Solution: Differentiating $x' = 3x - 4y$ gives $x'' = 3x' - 4y'$. But $y' = -2x + y$, so

$$\begin{aligned} x'' &= 3x' - 4(-2x + y) \\ &= 3x' + 8x - 4y \\ &= 3x' + 8x + (x' - 3x) \quad (\text{since } -4y = x' - 3x) \\ &= 4x' + 5x\,. \end{aligned}$$

That is, $x'' - 4x' - 5x = 0$.

For $x'' - 4x' - 5x = 0$, the auxiliary equation is $m^2 - 4m - 5 = 0$ with roots $m = 5$ or $m = -1$.
So $x = Ae^{5t} + Be^{-t}$. Hence $x' = 5Ae^{5t} - Be^{-t}$, and from $x' = 3x - 4y$ we obtain

$$\begin{aligned} y &= \frac{1}{4}(3x - x') \\ &= \frac{1}{4}[3(Ae^{5t} + Be^{-t}) - (5Ae^{5t} - Be^{-t})] \\ &= -\frac{1}{2}Ae^{5t} + Be^{-t}\,. \end{aligned}$$

2. Two species are in a predator-prey relationship. One species, which numbers Y individuals, eats the other, which numbers X individuals. Historically the numbers of these species have been constant with $X = 3000$ and $Y = 1500$. After a severe environmental disturbance the populations cease to be constant and start to change with time.

Let $x(t)$ and $y(t)$ be the difference between the historically constant population numbers and the new, changing population numbers $X(t)$ and $Y(t)$.
Then $x(t) = X(t) - 3000$ and $y(t) = Y(t) - 1500$ are the sizes of the perturbations from the historically steady states.
The sizes of the perturbations are described by the following:

$$\begin{aligned} x'(t) &= 3x(t) - 2y(t) \\ y'(t) &= 4x(t) - y(t). \end{aligned}$$

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(a) Show that $x''(t) - 2x'(t) + 5x(t) = 0$.

Solution:

$$\begin{aligned} x' &= 3x - 2y \\ x'' &= 3x' - 2y' \\ &= 3x' - 2(4x - y) \\ &= 3x' - 8x + 2y \\ &= 3x' - 8x + (3x - x') \\ &= 2x' - 5x\,. \end{aligned}$$

That is, $x'' - 2x' + 5x = 0$.

(b) Find $x(t)$ if $x(0) = 100$ and $x'(0) = 100$. (Take $t = 0$ to be the time at which monitoring of the population sizes begins.)

Solution: The auxiliary equation is $m^2 - 2m + 5 = 0$, with roots $m = 1 + 2i$ and $m = 1 - 2i$.
Therefore $x = e^t\,(A\cos 2t + B\sin 2t)\,.$
If $x = 100$ when $t = 0$, then $A = 100$ and $x = e^t(100\cos 2t + B\sin 2t)$.
Hence, $x' = e^t(-200\sin 2t + 2B\cos 2t) + e^t(100\cos 2t + B\sin 2t)$. When $t = 0$, $x' = 2B + 100 = 100$, and so $B = 0$. Therefore $x = 100e^t\cos 2t$.

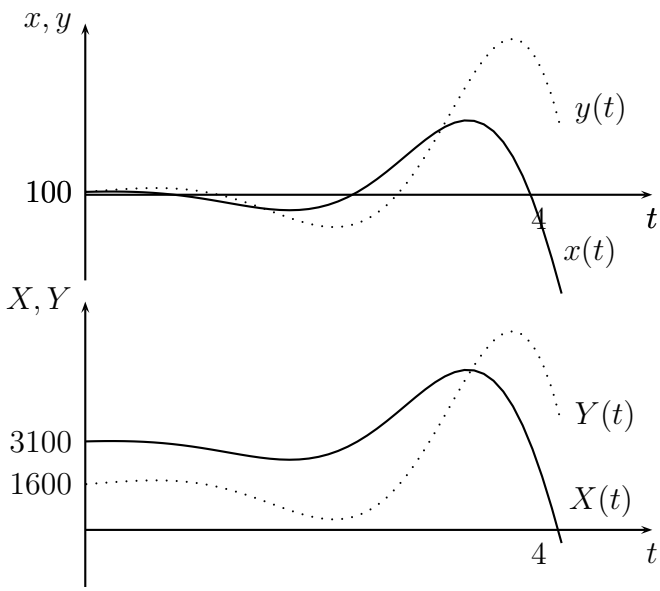
(c) Hence find $y(t)$.

Solution:

$$\begin{aligned} y(t) &= \frac{1}{2}(3x - x') \\ &= \frac{1}{2}(300e^t\cos 2t - 100e^t\cos 2t + 200e^t\sin 2t) \\ &= 100e^t(\cos 2t + \sin 2t)\,. \end{aligned}$$

(d) Sketch $x(t)$ and $y(t)$ and then $X(t)$ and $Y(t)$ as a function of t. What will happen to the original populations?

Solution:



The populations will oscillate with increasingly large swings. Eventually the swings will get so big that $X(t) = 3000 + x(t)$ will be zero. (Once this happens the predictions of the model will no longer be valid.) The prey will then become extinct. The predator will then also become extinct unless it has an alternative source of food.

3. Find whether each of the following first-order equations is separable, linear or neither. If the equation is separable or linear find its general solution.

(a) $\dfrac{dy}{dx} = \dfrac{x^2 - y}{x}$

Solution: The equation is not separable, however we can rearrange the equation as

$$\frac{dy}{dx} + \frac{y}{x} = x,$$

which we now recognize as first-order linear. The integrating factor is then

$$r(x) = e^{\int (1/x)dx} = e^{\ln x} = x.$$

Multiplying the equation by $r(x)$, we obtain

$$x\frac{dy}{dx} + y = x^2 \text{ or } \frac{d}{dx}(xy) = x^2.$$

Integrating both sides with respect to x, we get

$$xy = \frac{x^3}{3} + C.$$

Thus we obtain the general solution $y = \dfrac{x^2}{3} + \dfrac{C}{x}$.

(b) $\dfrac{dy}{dx} = \dfrac{y-1}{x(1+x)}$

Solution: The equation

$$\frac{dy}{dx} = \frac{y-1}{x(1+x)} = (y-1)\left(\frac{1}{x(1+x)}\right)$$

is separable, so we separate and integrate:

$$\int \frac{dy}{y-1} = \int \frac{dx}{x(1+x)} = \int \left(\frac{1}{x} - \frac{1}{1+x}\right) dx \quad \text{(by partial fractions).}$$

So

$$\ln|y-1| = \ln|x| - \ln|1+x| + C = \ln\left|\frac{x}{1+x}\right| + C.$$

Therefore,

$$y - 1 = \frac{Ax}{1+x}, \text{ where } A = \pm e^C,$$

and so the general solution is $y = \dfrac{Ax}{1+x} + 1$.

(c) $\dfrac{dy}{dx} = \dfrac{\sin x - y}{x - \cos y}$

Solution: This equation is neither separable nor linear.

(d) $\dfrac{dy}{dx} = \dfrac{y^2 - 1}{2(1+x)y}$

Solution: The equation

$$\frac{dy}{dx} = \frac{y^2 - 1}{2(1+x)y} = \left(\frac{y^2 - 1}{2y}\right)\left(\frac{1}{1+x}\right)$$

is separable, so separate and integrate:

$$\int \frac{2y}{y^2 - 1} dy = \int \frac{dx}{1+x}.$$

This gives $\ln|y^2 - 1| = \ln|1+x| + C$, and so $y^2 - 1 = A(1+x)$, where $A = \pm e^C$. Thus we obtain the general solution $y^2 = A(1+x) + 1 = Ax + (A+1)$.