1. The alphabet of a certain language consists of there letters a, b, c. Any string of the letters is a word in the language.

(a) What is the number of words of length 6? (*2 marks*)

(b) In how many of the words of length 6 the last three letters are all different? (*2 marks*)

(c) How many of the words of length 6 contain exactly three a's? (*2 marks*)

(d) How many of the words of length 6 commence with aa or end with cc? (*2 marks*)

(e) In how many of the words of length 6 the letter a never follows b or c, and b never follows c? (*2 marks*)

Solution

(a) This is the number of ordered selections of 6 things from 3 things with repetitions allowed. So, the number is $3^6 = 729$.

(b) There are $3! = 6$ ways to choose the last three letters. The number of choices for the first three letters is $3^3 = 27$. By the multiplication principle, the total number is $6 \cdot 27 = 162$.

(c) There are $\binom{6}{3} = 20$ ways to choose 3 places for the a's. The remaining 3 places can be filled in $2^3 = 8$ ways. By the multiplication principle, the total number is $20 \cdot 8 = 160$.

(d) The number of words commencing with aa is $3^4 = 81$, and the number of words ending with cc is $3^4 = 81$. The required number is $81 + 81 - 9 = 153$, where $9 = 3^2$ counts the words which commence with aa and end with cc.

(e) Such words can only have the form: a few a's followed by a few b's and followed by a few c's. The number of the words is therefore the number of unordered selections of 6 things from 3 things with repetitions allowed. That is, $\binom{6+3-1}{6} = \binom{8}{6} = \binom{8}{2} = 28$.

2. Twenty oranges are to be put in ten labelled boxes.

(a) What is the number of ways to do this, if the oranges are indistinguishable? (*2 marks*)

(b) What is the number of ways to do this, if the oranges are indistinguishable and exactly five of the boxes should be empty? (*3 marks*)

(c) What is the number of ways to do this if the oranges are labelled? (*2 marks*)

(d) What is the number of ways to do this if the oranges are labelled and each box contains two oranges? (*3 marks*)

Solution

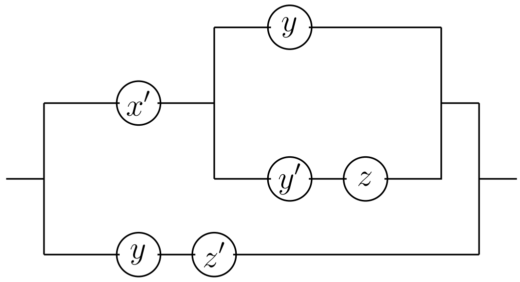
(a) This is the number of unordered selections of 20 things out of 10 things with repetitions allowed, that is, $\binom{10+20-1}{20} = \binom{29}{9}$.

(b) The number of ways to choose 5 boxes is $\binom{10}{5}$. Place one orange in each of the remaining 5 boxes. Now we need to count unordered selections of 15 things out of 5 things with repetitions allowed, that is, $\binom{15+5-1}{15} = \binom{19}{4}$. The total number is $\binom{10}{5} \cdot \binom{19}{4}$.

(c) This is the number of ordered selections of 20 things out of 10 things with repetitions allowed, that is, 10^{20}.

(d) This is the number of arrangements and so given by the multinomial coefficient $\binom{20}{2,2,2,2,2,2,2,2,2,2} = \frac{20!}{1024}$.

3. A Boolean function f in three variables x, y, z is represented by the switching circuit



(a) Write down the Boolean expression representing the function f. *(2 marks)*

(b) Complete the table of values for f. *(2 marks)*

(c) Write down the disjunctive normal form for f. *(2 marks)*

(d) Apply the Karnaugh map method to find a simpler Boolean expression for f. *(2 marks)*

(e) Hence draw a simpler switching circuit and a digital logic circuit representing the function f. *(2 marks)*

Solution

(a) The Boolean expression is $x'(y \vee y'z) \vee yz'$.

(b) The table of values is

x	y	z	f
1	1	1	0
1	1	0	1
1	0	1	0
1	0	0	0
0	1	1	1
0	1	0	1
0	0	1	1
0	0	0	0

(c) The disjunctive normal form is

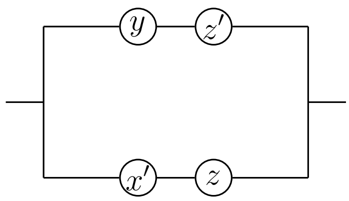
$$f = xyz' \vee x'yz \vee x'yz' \vee x'y'z.$$

(d) The corresponding Karnaugh map is

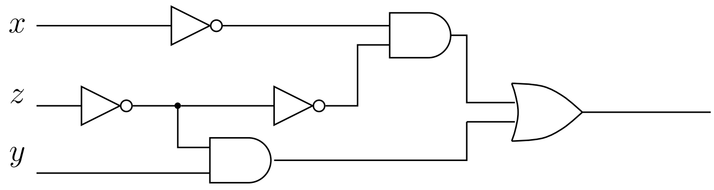
	yz	yz'	$y'z'$	$y'z$
x	0	1	0	0
x'	1	1	0	1

The 1's in the map are covered by two blocks of two 1's thus producing a simpler Boolean expression $x'z \vee yz'$.

(e) A switching circuit representing the function f is



A digital logic circuit representing the function f is



4. The generating function of a sequence $\{x_n\}$, $n \geq 0$, is given by the formula

$$G(z) = \frac{1}{1-2z} + \frac{c}{1+2z},$$

where c is a real number. Different values of c correspond to different sequences $\{x_n\}$.

(a) Find a value of c such that the corresponding sequence $\{x_n\}$ satisfies the initial conditions $x_0 = 4$ and $x_1 = -4$. *(3 marks)*

(b) Find a value of c for which the corresponding sequence $\{x_n\}$ satisfies the recurrence relation $x_{n+2} = -x_{n+1} + 6\,x_n, \quad n \geq 0.$ *(4 marks)*

(c) Find the general solution of the recurrence relation in part (b). *(3 marks)*

Solution

(a) The sequence $\{x_n\}$ is given by $x_n = 2^n + c\,(-2)^n$. We have $1 + c = 4$ and $2 - 2c = -4$. Hence, $c = 3$.

(b) Substitute $x_n = 2^n + c\,(-2)^n$ into the relation:

$$2^{n+2} + c\,(-2)^{n+2} + 2^{n+1} + c\,(-2)^{n+1} - 6 \cdot 2^n - 6\,c\,(-2)^n = 0$$

for $n \geq 0$. Taking $n = 0$ we get $-4\,c = 0$, hence $c = 0$.

(c) The characteristic equation is $\lambda^2 + \lambda - 6 = 0$ which has roots 2 and -3. Therefore, the general solution is $x_n = A\,2^n + B\,(-3)^n$.

5. Use mathematical induction to prove that

(a) $3^n + 19^n$ is divisible by 22 for any positive odd integer n. (*3 marks*)

(b) $7^n \geq 6^{n-1}(n+6)$ for any integer $n \geq 1$. (*3 marks*)

(c) $\binom{n}{m} - \binom{n}{m-1} + \binom{n}{m-2} - \cdots + (-1)^m \binom{n}{0} = \binom{n-1}{m}$ for $0 \leq m < n$. (*4 marks*)

Solution

(a) Let $S(n)$ be the statement. Then $S(1)$ is true since $3^1 + 19^1 = 22$ is divisible by 22. Suppose that $S(n)$ holds for an odd $n \geq 1$. Then

$$3^{n+2} + 19^{n+2} = 9 \cdot 3^n + 361 \cdot 19^n = 9(3^n + 19^n) + 352 \cdot 19^n.$$

Since $352 = 22 \cdot 16$, by the induction hypothesis the sum is divisible by 22. Hence, $S(n+2)$ holds. Thus, $S(n)$ is true for all odd positive n.

(b) Let $S(n)$ be the proposition $7^n \geq 6^{n-1}(n+6)$. For $n = 1$ we have $7 \geq 6^0(1+6)$, that is, $7 \geq 7$ which is true. Suppose that $S(n)$ is true for some $n \geq 1$, i.e., $7^n \geq 6^{n-1}(n+6)$. We have

$$\begin{aligned} 7^{n+1} &= 7 \cdot 7^n \geq 7 \cdot 6^{n-1}(n+6) = (6+1) \cdot 6^{n-1}(n+6) \\ &= 6^n(n+6) + 6^{n-1}(n+6) \geq 6^n(n+6) + 6^n = 6^n(n+7), \end{aligned}$$

and so, $S(n+1)$ is true. Hence $S(n)$ is true for all $n \geq 1$.

(c) Use induction on m keeping n fixed. Let $S(m)$ be the proposition. For $m = 0$ we have $\binom{n}{0} = \binom{n-1}{0}$ and so $S(0)$ is true. Suppose that $S(m)$ is true for some $0 \leq m < n-1$. By the hypothesis we have

$$\sum_{k=0}^{m+1} (-1)^{m-k+1} \binom{n}{k} = -\binom{n-1}{m} + \binom{n}{m+1}.$$

However, by the recurrence relation for the binomial coefficients

$$-\binom{n-1}{m} + \binom{n}{m+1} = \binom{n-1}{m+1}.$$

Hence $S(m+1)$ is true. Therefore $S(m)$ is true for all m such that $0 \leq m < n$.

1. Consider the strings of length 5 formed by the digits 1, 2, 3, 4, 5, 6, 7.

(a) What is the total number of such strings? (*2 marks*)

(b) In how many of the strings all digits are distinct? (*2 marks*)

(c) In how many of the strings the digits are strictly increasing (i.e. each digit is greater than the previous one)? (*2 marks*)

(d) In how many of all strings the digits are increasing (i.e. each digit is not greater than the next one)? (*2 marks*)

(e) How many of the strings in (b) do not contain the string "1234"? (*2 marks*)

Solution

(a) This is the number of ordered selections of 5 things from 7 things with repetitions allowed. So, the number is $7^5 = 16807$.

(b) This is the number of ordered selections of 5 things from 7 things without repetitions. So, the number is $7_{(5)} = \frac{7!}{2!} = 2520$.

(c) This is equal to the number of unordered selections of 5 things from 7 things without repetitions. So, the number is $\binom{7}{5} = \frac{7!}{2!5!} = 21$.

(d) This is equal to the number of unordered selections of 5 things from 7 things with repetitions allowed. So, the number is $\binom{7+5-1}{5} = \binom{11}{5} = \frac{11!}{5!6!} = 462$.

(e) Let us count first how many of the strings contain the substring "1234". If this substring occurs in the beginning of the string, there are 3 ways to complete it to the 5-digit string, as repetitions are not allowed. If the substring "1234" occurs at the end of the string then there are also 3 ways to complete it to the 5-digit string. So in total, 6 strings contain the substring "1234". Therefore, the answer is $2520 - 6 = 2514$.

2. (a) How many arrangements are there of the letters
R E P E T I T I V E ? (*2 marks*)

(b) How many arrangements are there with the three Es together? (*2 marks*)

(c) How many arrangements are there with the two Is together and the two Ts together? (*3 marks*)

(d) How many arrangements are there with the R, P and V all next to one another? (*3 marks*)

Solution

(a) There are three Es, 2 Ts, 2 Is, 1 P, 1 R and 1 V. So

$$\binom{10}{3,2,2,1,1,1} = \frac{10!}{3!\,2!\,2!} = 151200$$

arrangements.

(b) Treat the three Es as one unit. There are then

$$\binom{8}{2,2,1,1,1,1} = \frac{8!}{2!\,2!} = 10080$$

arrangements.

(c) Treat the two Is as one unit and the two Ts as one unit. There are then

$$\binom{8}{3,1,1,1,1,1} = \frac{8!}{3!} = 6720$$

arrangements.

(d) We treat the R, P and V as one unit, but within that unit there are $3! = 6$ possible arrangements of the three letters. So there are

$$\binom{8}{3,2,2} \times 3! = \frac{8!}{3!\,2!\,2!} \times 3! = \frac{8!}{2!\,2!} = 10080$$

arrangements.

3. For propositions x, y and z consider the compound proposition f given by $(x \Rightarrow y) \Rightarrow z$.

(a) Construct the truth table for the proposition f. *(2 marks)*

(b) Interpreting f as a Boolean function in the variables x, y and z (replace T by 1 and F by 0), write down the Boolean expression for f in the disjunctive normal form. *(2 marks)*

(c) Apply the Karnaugh map method to find a simpler Boolean expression for f. *(2 marks)*

(d) Using the Boolean expression you found in (c) draw

(*i*) A switching circuit representing the function f. *(2 marks)*

(*ii*) A digital logic circuit representing the function f. *(2 marks)*

Solution

(a) The truth table is

x	y	z	f
T	T	T	T
T	T	F	F
T	F	T	T
T	F	F	T
F	T	T	T
F	T	F	F
F	F	T	T
F	F	F	F

(b) The Boolean expression in the disjunctive normal form is

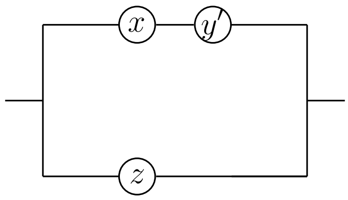
$$f = xyz \vee xy'z \vee xy'z' \vee x'yz \vee x'y'z.$$

(c) The corresponding Karnaugh map is

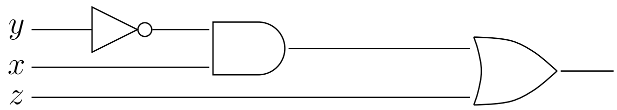
	yz	yz'	$y'z'$	$y'z$
x	1	0	1	1
x'	1	0	0	1

The 1's in the map are covered by two blocks of two 1's and four 1's thus producing a simpler Boolean expression $xy' \vee z$.

(d) (i) A switching circuit representing the function f is



(ii) A digital logic circuit representing the function f is



4. (a) Find the general solutions of the recurrence relation
$x_n = -9x_{n-1} - 14x_{n-2}, \quad n \geq 2.$ (*3 marks*)

(b) Find the solution of the recurrence relation in (a) satisfying the initial conditions $x_0 = 5$ and $x_1 = -45$. (*3 marks*)

(c) Write down the generating function of the sequence $\{x_n\}$ you found in (b). (*4 marks*)

Solution

(a) The characteristic equation is $\lambda^2 + 9\lambda + 14 = 0$ which has roots -2 and -7. Therefore, the general solution is $x_n = A\,(-2)^n + B\,(-7)^n$.

(b) The initial conditions give $A + B = 5$ and $-2A - 7B = -45$ and so $A = -2$, $B = 7$. Hence, $x_n = -2 \cdot (-2)^n + 7 \cdot (-7)^n$.

(c) The generating function is

$$G(z) = \sum_{n=0}^{\infty} x_n\, z^n = -2\sum_{n=0}^{\infty}(-2)^n z^n + 7\sum_{n=0}^{\infty}(-7)^n z^n = \frac{7}{1+7z} - \frac{2}{1+2z} = \frac{5}{1+9z+14z^2}.$$

5. Let b_n, $n \geq 0$, be the number of solutions of the equation

$$x + 2\,y = n,$$

where x and y are nonnegative integers.

(a) Show that $b_n = b_{n-1}$ if n is odd, and $b_n = b_{n-1} + 1$ if n is even. (*3 marks*)

(b) Using (a) or otherwise, find a closed form for the generating function of the sequence $\{b_n\}$. (*4 marks*)

(c) Show that b_n satisfies the relation
$b_n + b_{n-1} = n + 1$. *(3 marks)*

Solution

(a) If $n = 2k$ is even then the solutions are $(2k, 0), (2k-2, 1), \dots, (0, k)$ and so $b_{2k} = k+1$. If $n = 2k+1$ is odd then the solutions are $(2k+1, 0), (2k-1, 1), \dots, (1, k)$ and so $b_{2k+1} = k+1$. Hence, $b_{2k+1} = b_{2k}$ and $b_{2k} = b_{2k-1} + 1$.

(b) We have

$$G(z) - zG(z) = b_0 + (b_1 - b_0)z + (b_2 - b_1)z^2 + \cdots = 1 + z^2 + z^4 + \cdots = \frac{1}{1-z^2}.$$

Hence,

$$G(z) = \frac{1}{(1-z)(1-z^2)}.$$

(c) We have

$$(1+z)G(z) = \frac{1}{(1-z)^2} = \sum_{n=0}^{\infty}(n+1)z^n.$$

Hence, $b_n + b_{n-1} = n + 1$.