THE UNIVERSITY OF SYDNEY

MATH 1004

Second Semester	Discrete Mathematics	2011

Tutorial 1 Week 2

1. Which of the following strings of brackets are balanced? In each case, explain carefully why the string is, or is not, balanced:

(*i*) ((()()

(*ii*) (())()

(*iii*) ())(()

Solution.

(*i*) Since there are more left brackets than right brackets, we see that the given string of brackets is not balanced.

(*ii*) Since there are two left brackets followed by two right brackets and then one left followed by one right, the given string of brackets is balanced.

(*iii*) Since there is one left bracket followed by two right brackets, the given string of brackets is not balanced.

2. Suppose you have numbers x_0, x_1, x_2, ... , x_n and you want to multiply them together. In how many ways can you insert brackets into the string $x_0x_1x_2 \dots x_n$ so that the order of multiplication is completely specified? Each pair of brackets should contain just two terms. For example, when $n = 2$, there are two ways: $(x_0(x_1x_2))$ and $((x_0x_1)x_2)$. Try $n = 3$ and $n = 4$.

Solution.

For $n = 3$, consider x_0, x_1, x_2, x_3. There are five ways:

$$(x_0(x_1(x_2x_3))); \ (x_0((x_1x_2)x_3)); \ ((x_0(x_1x_2))x_3);$$

$$(((x_0x_1)x_2)x_3), \ ((x_0x_1)(x_2x_3)).$$

For $n = 4$, consider x_0, x_1, x_2, x_3, x_4. There are 14 ways:

$$(x_0(x_1(x_2(x_3x_4)))); \ (x_0(x_1((x_2x_3)x_4))); \ (x_0((x_1(x_2x_3))x_4));$$

$$(x_0((x_1x_2)(x_3x_4))); \ ((x_0(x_1x_2))(x_3x_4)); \ (((x_0x_1)x_2)(x_3x_4));$$

$$((x_0x_1)(x_2(x_3x_4))); \ ((x_0(x_1(x_2x_3)))x_4); \ (x_0(((x_1x_2)x_3)x_4));$$

$$((x_0x_1)((x_2x_3)x_4)); \ (((x_0x_1)(x_2x_3))x_4); \ ((x_0((x_1x_2)x_3))x_4);$$

$$(((x_0(x_1x_2))x_3)x_4); \ ((((x_0x_1)x_2)x_3)x_4).$$

3. Consider a river system with n sources which eventually merge to form a single stream. Assuming that no more than two streams merge at any point, we are interested in the number of ways that the mergers can take place.

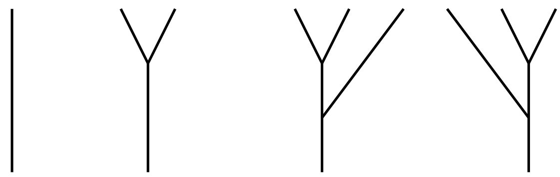
(*i*) Compile a table of values for $n = 1$, 2, 3, and 4 and then find (guess) a general formula.

(*ii*) If possible, find a connection with the bracketing problem in Question 2.

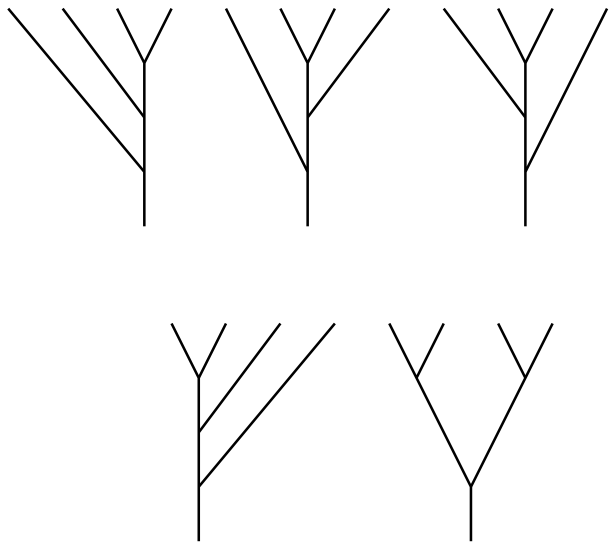
Solution.

(*i*) For $n = 1, 2, 3, 4$, we sketch all the possible cases as follows:

$n = 1$ $n = 2$ $n = 3$



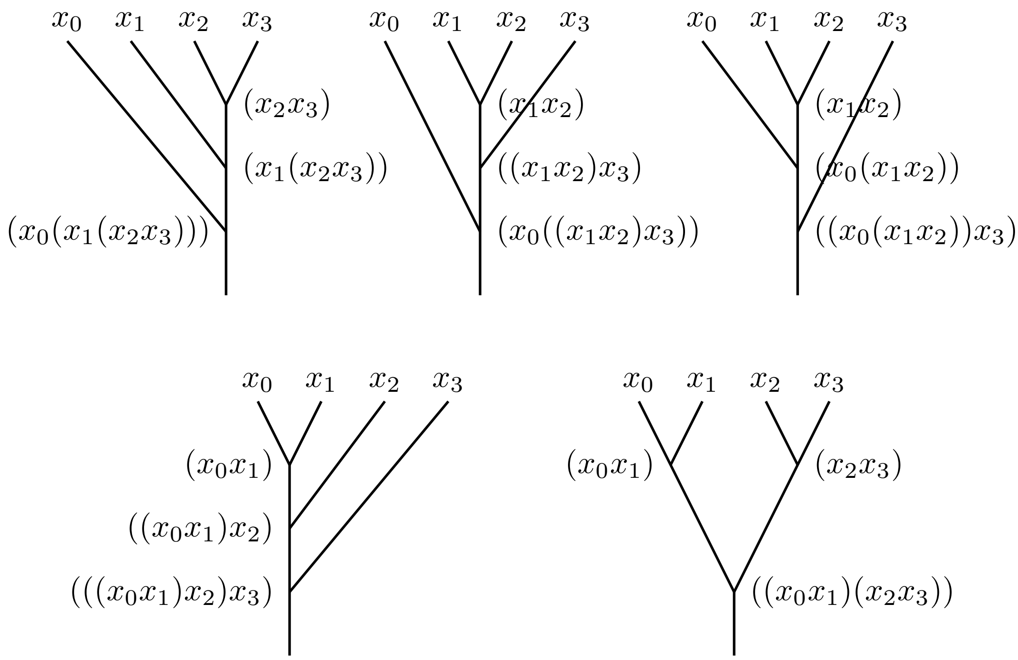
$n = 4$



Hence for $n = 1$, the number of ways that the mergers can take place is 1, for $n = 2$, the number of ways is 1, for $n = 3$, the number of ways is 2 and for $n = 4$, the number of ways is 5.

For a river system with n sources, the number of ways that the mergers can take place is c_{n-1}, the $(n-1)^{\text{st}}$ Catalan number.

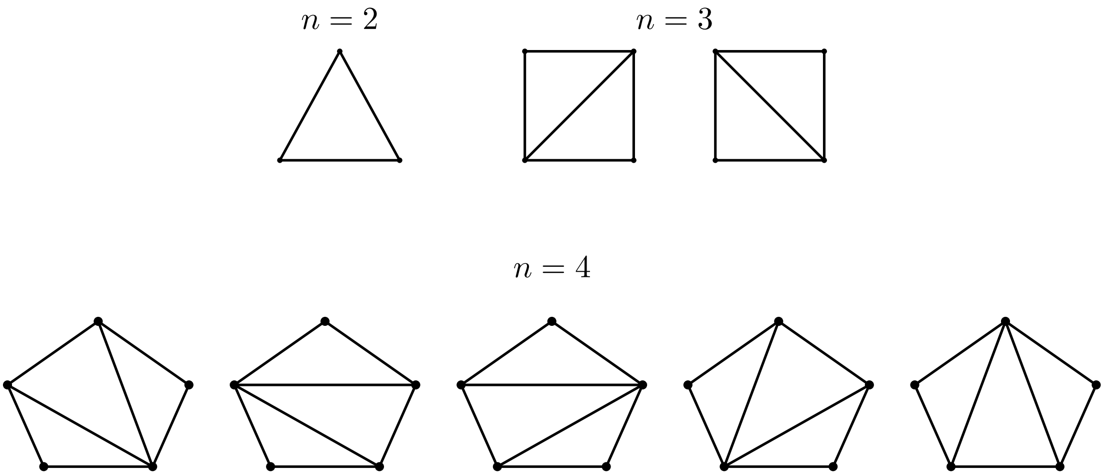
(*ii*) Let the n sources of the river system be $x_0, x_1, \ldots, x_{n-1}$. When two streams merge at a point, we put a bracket around them. This is a one-to-one correspondence between mergers and bracketings of $x_0, x_1, \ldots, x_{n-1}$. For example, for $n = 4$, the four sources are denoted by x_0, x_1, x_2, x_3, and the following gives a connection between the two problems:



4. In how many ways can a convex polygon with $n+1$ sides (labelled 0, 1, 2, . . . , n) be divided into triangles by non-intersecting diagonals? If possible, find a connection with the bracketing problem in Question 2.

Solution.

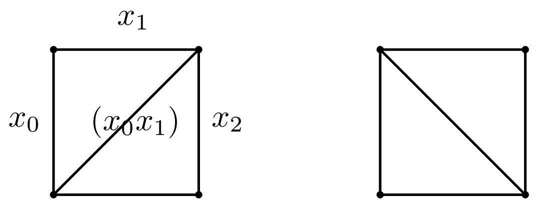
For $n = 2, 3, 4$, the dissections are drawn as follows:



Hence, for $n = 2, 3$ and 4, the number of ways of dissecting are 1, 2 and 5 respectively.

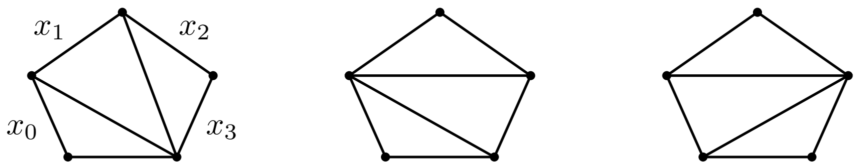
A connection with the bracketing problem in Question 2. is described as follows. For $i = 0, 1, \dots, n-1$, label side i with x_i. For each triangle, if two sides are labelled, then label the third side with a bracket containing the product of the other two sides. Thus, for $n = 3$ and $n = 4$, the following gives the connection between the two problems

$n = 3$



$((x_0x_1)x_2)$ $\quad$ $(x_0(x_1x_2))$

$n = 4$



$(x_0(x_1(x_2x_3)))$ $\quad$ $(x_0((x_1x_2)x_3))$ $\quad$ $((x_0(x_1x_2))x_3)$

$(((x_0x_1)x_2)x_3)$ $\quad$ $((x_0x_1)(x_2x_3))$

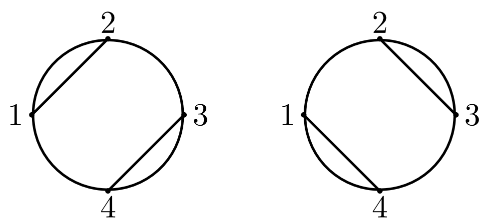
5. For $n \geq 0$, evenly distribute $2n$ points on the circumference of a circle. Let a_n be the number of ways in which these $2n$ points can be paired off as n chords where no two chords intersect.

(i) Find a_n for $n = 2$, 3 and then find (or guess) a general formula.

(ii) If possible, find a connection with the first bracket problem [Problem 1.1].

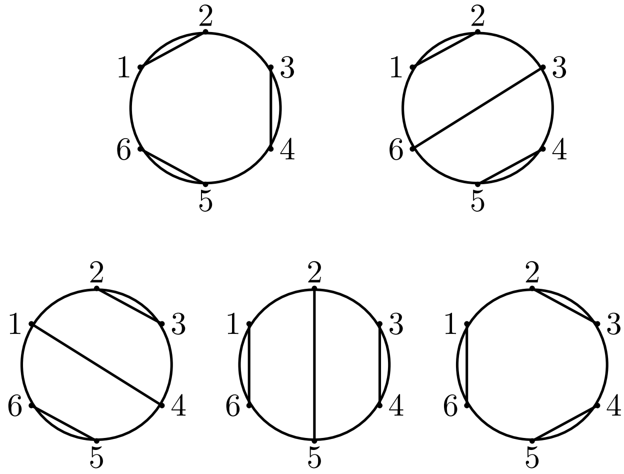
Solution.

(i) For $n = 2$, the possible ways in which 4 points can be paired off as 2 chords where no two chords intersect are as shown below:



Hence we see that $a_2 = 2$.

For $n = 3$, the possible ways in which 6 points can be paired off as 3 chords where no two chords intersect are as shown below:

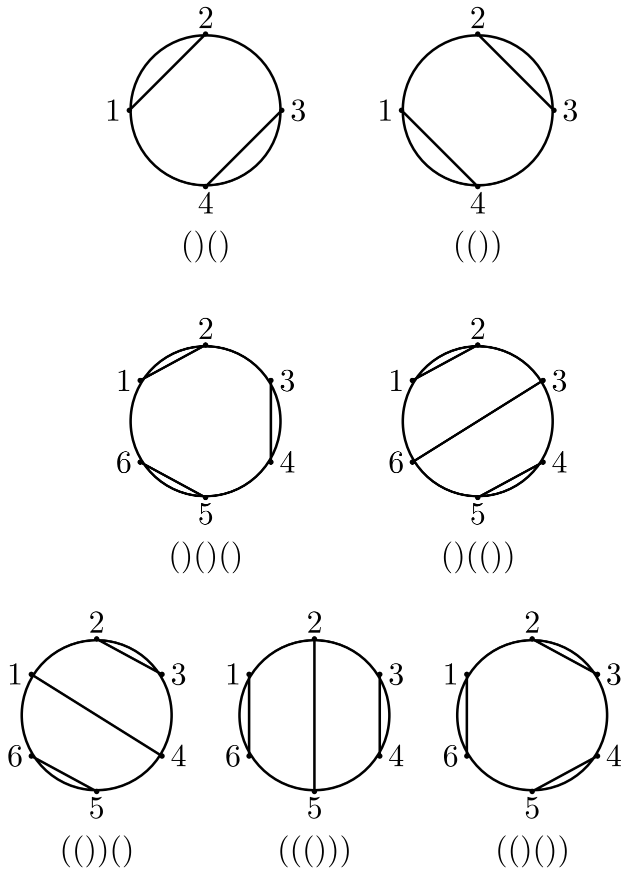


Hence we see that $a_3 = 5$.

In general, we guess that a_n is just the n^{th} Catalan number c_n.

(*ii*) Label the $2n$ points with the numbers 1, 2, … , $2n$. If for $i < j$, point i is joined to point j, then in the corresponding strings of n pairs of brackets, there is a left bracket in the i-th position and a right bracket in the j-th position.

For $n = 2$ and $n = 3$, the following gives the connection between the given problem and Problem 1.1:



6. For each of the following balanced strings of brackets,

(*i*) LLRLRRLLRRLRLR (*ii*) LLRLRLRRLRLLRRLLRLRR;

(a) draw the corresponding planar diagram;

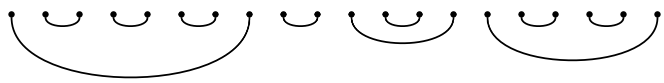
(b) construct the corresponding way of pairing off $2n$ points as n chords where no two chords intersect.

Solution.

(a) Joining each L with its matching R, then replacing Ls and Rs by dots, gives the corresponding planar diagram:

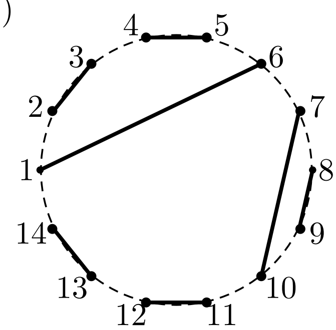
(*i*)

(*ii*)

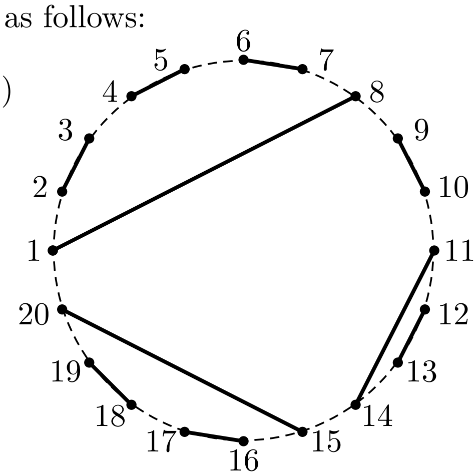


(b) The corresponding ways of pairing are as follows:

(*i*)



(*ii*)



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Problem Set 1

1. (*i*) Which of the following strings of brackets are balanced? In each case, explain carefully why the string is, or is not, balanced:

(a) (()()() (b) (())(()()) (c) (()))(

(*ii*) Construct a balanced string of brackets corresponding to the following planar diagram or smiling face:

(*iii*) Construct the planar diagram or smiling face corresponding to the following balanced string of brackets (()()()))()(())(()()).

Solution.

(*i*) (a) The string is not balanced, since there are more left brackets then right brackets. [Or: since the first left bracket has no matching right bracket.]

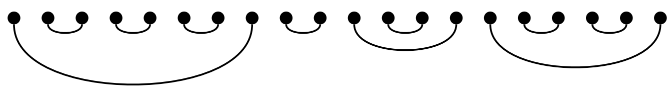
(b) The string is balanced, since each left bracket has a matching right bracket.

(c) The string is not balanced, since the last left bracket has no matching right bracket.

(*ii*) The corresponding balanced string of brackets is

(())(()())() or LLRRLLRLRRLR.

(*iii*) The corrresponding planar diagram is



2. Given two rows of boxes with n boxes in each row:

		$\cdots$	
		$\cdots$	

In how many ways can you place the numbers 1, 2, ... , $2n$ in the boxes so that the numbers increase from left to right and so that each number in the bottom row is larger than the number in the box above it? Write down all the arrangements for $n = 1$, 2, 3 and 4. Any conjectures?

Solution.

Let a_n be the number of ways to place the numbers 1, 2, ... , $2n$ in the boxes with the given conditions.

For $n = 1$, there is only one such arrangement and for $n = 2$, there are 2 such possible arrangements:

1
2

1	2
3	4

1	3
2	4

Thus $a_1 = 1$ and $a_2 = 2$.

For $n = 3$, we have the following possible arrangements:

1	2	3
4	5	6

1	2	4
3	5	6

1	2	5
3	4	6

1	3	4
2	5	6

1	3	5
2	4	6

and therefore $a_3 = 5$.

When $n = 4$, the possible ways of arrangements are:

1	2	3	4
5	6	7	8

1	2	3	5
4	6	7	8

1	2	3	6
4	5	7	8

1	2	3	7
4	5	6	8

1	2	4	5
3	6	7	8

1	2	4	6
3	5	7	8

1	2	4	7
3	5	6	8

1	2	5	6
3	4	7	8

1	2	5	7
3	4	6	8

1	3	4	5
2	6	7	8

1	3	4	6
2	5	7	8

1	3	4	7
2	5	6	8

1	3	5	6
2	4	7	8

1	3	5	7
2	4	6	8

and hence $a_4 = 14$. The numbers a_n seem to be the Catalan numbers.

3. For the following balanced string of brackets

LLRLRRLLRRLRLR,

(*i*) construct the corresponding planar diagram or smiling face;

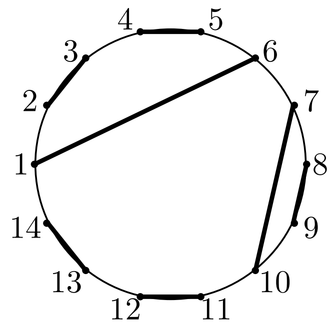
(*ii*) construct the corresponding way of pairing off 14 points on the circumference of a circle, as 7 chords where no two chords intersect. [i.e., the hand-shaking problem for 14 people!]

(*iii*) construct the corresponding $(7, 7)$ tableau; that is, given 2 rows of boxes with 7 boxes in each row, place the numbers $1, 2, \cdots, 14$ in the boxes so that the numbers increase from left to right and so that each number in the bottom row is larger than the number in the box above it.

Solution.

(*i*) The corrresponding planar diagram or smiling face is

(*ii*) The corresponding way of pairing off the 14 points is



(*iii*) The corresponding $(7, 7)$ tableau is

1	2	4	7	8	11	13
3	5	6	9	10	12	14

(For each $n = 1, 2, \ldots, 14$ put the number n in the first empty box in the top row if the nth bracket is an L, and in first empty box of the bottom row if the nth bracket is an R. The fact that there will then be 7 numbers in each row corresponds to the fact that there are 7 Ls and 7 Rs, and the fact that each number in the bottom row will then be greater than the number in the box above corresponds to the fact that the Ls and Rs are correctly nested. The fact that the numbers in each row increase from left to right is a trivial consequence of the construction, and would have been true even if the brackets had not been correctly nested.)

4. In a soccer match between Teams A and B, the final score is n−all. At no time during the match was Team B in the lead. In how many different ways can the A-B scores be built up, starting at 0-0 and ending at n-n? Try the cases $n = 1, 2, 3$ and write down the scoring patterns. Can you guess what the answer might be when $n = 4$? Find a connection between the scoring patterns and Problem 1.1.

Solution.

The possible scoring patterns for $n = 1, 2, 3$ are as follows: (Left hand numbers represent A's score, right hand numbers represent B's score.)

$n = 1$: 0-0,1-0,1-1.

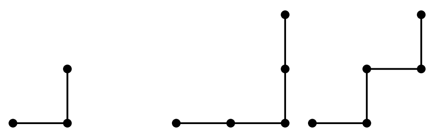
$n = 2$: 0-0,1-0,1-1,2-1,2-2 and 0-0,1-0,2-0,2-1,2-2.

$n = 3$: 0-0,1-0,1-1,2-1,3-1,3-2,3-3; 0-0,1-0,1-1,2-1,2-2,3-2,3-3 ;
0-0,1-0,2-0,2-1,2-2,3-2,3-3; 0-0,1-0,2-0,2-1,3-1,3-2,3-3
and 0-0,1-0,2-0,3-0,3-1,3-2,3-3.

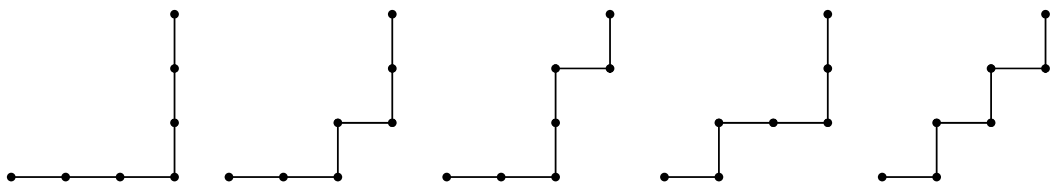
Hence there is 1 scoring pattern when $n = 1$, 2 when $n = 2$, and 5 when $n = 3$.

When $n = 4$ there are 14 scoring patterns, as the number of scoring patterns, for any n, is the Catalan number c_n.

A good way of illustrating the scoring patterns is by plotting a path between points with integer coefficients in the xy plane. For example, for n = 1, 2, 3, the scoring patterns are represented as follows:



$n = 1$ $n = 2$



$n = 3$

The connection between this problem and Problem 1.1 is as follows. Associate a left bracket "(" with every unit move to the right, and a right bracket ")" with every unit move upwards.