The University of Sydney

MATH 1004

Second Semester	Discrete Mathematics	2011

Tutorial 2 Week 3

1. Use the notation of set theory to describe:

(i) The set of all odd integers between 2 and 10.

(ii) The set of all odd integers between 2 and 200.

(iii) The set of all odd integers.

(iv) The set of integers divisible by 4.

For this exercise, do *not* use the dots notation (like ...).

Solution.

Let $\mathbb{Z}$ be the set of all integers.

(i) The set of all odd integers between 2 and 10 is $\{3, 5, 7, 9\}$

(ii) The set of all odd integers between 2 and 200 is

$$\{x \mid x = 2y + 1,\ y \in \mathbb{Z},\ 1 \le y \le 99\}.$$

(iii) The set of all odd integers is $\{x \mid x = 2y + 1,\ y \in \mathbb{Z}\}$

(iv) The set of integers divisible by 4 is $\{x \mid x = 4y,\ y \in \mathbb{Z}\}$

2. Which of the following statements are true?

(i) $\{2, 4\} \subseteq \{1, 2, 3, 4, 5, 6\}$.

(ii) $\{2\} \subseteq \{1, 2, 3, 4, 5, 6\}$.

(iii) $2 \subseteq \{1, 2, 3, 4, 5, 6\}$.

(iv) $2 \in \{1, 2, 3, 4, 5, 6\}$.

(v) $\{2\} \in \{1, 2, 3, 4, 5, 6\}$.

Give reasons for your answers.

Solution.

(i) The statement is true since the elements 2 and 4 in the set $\{2, 4\}$ are also in the set $\{1, 2, 3, 4, 5, 6\}$.

(ii) The statement is again true.

(iii) The statement is false; since 2 is an element, but not a subset.

(iv) The statement is true since 2 is an element in the set $\{1, 2, 3, 4, 5, 6\}$.

(v) The statement is false; since $\{2\}$ is a subset, but is not an element in the given set.

3. Let $A = \{1,\ 2,\ 3,\ \{2\},\ \{2,\ 3\},\ 4\}$. Which of the following statements are true?

(i) $\{2\} \in A$.

(ii) $\{\{2\}\} \subseteq A$

(iii) $\{2,\ \{2\}\} \subseteq A$.

(iv) $\{2,\ \{3\}\} \subseteq A$.

(v) $\{2,\ 3\} \in A$.

(vi) $\{3,\ \{2, 3\}\} \subseteq A$.

Solution.

(i) True, by definition.

(ii) True, since $\{2\}$ is in A.

(iii) True, since both 2 and $\{2\}$ are in A.

(iv) False, since $\{3\}$ is not in A.

(v) True, by definition.

(vi) True, since both 3 and $\{2,\ 3\}$ are in A.

4. Write out the following sets, where $A = \{a, b, c, \{a, d\}\}$:

(i) $A \cup \{b, d, e\}$.

(ii) $A \cap \{b, d, e\}$.

(iii) $A \setminus \{a, b\}$.

(iv) $A \setminus \{c, d\}$.

(v) $A \setminus \{\{a, d\}\}$.

(vi) $A \setminus \{a, \{a, d\}\}$.

Then write down the sizes of each of the sets.

Solution.

(i) $A_1 = A \cup \{b, d, e\} = \{a, b, c, d, e, \{a, d\}\}$ and $|A_1| = 6$.

(ii) $A_2 = A \cap \{b, d, e\} = \{b\}$ and $|A_2| = 1$.

(iii) $A_3 = A \setminus \{a, b\} = \{c, \{a, d\}\}$ and $|A_3| = 2$.

(iv) $A_4 = A \setminus \{c, d\} = \{a, b, \{a, d\}\}$ and $|A_4| = 3$.

(v) $A_5 = A \setminus \{\{a, d\}\} = \{a, b, c\}$ and $|A_5| = 3$.

(vi) $A_6 = A \setminus \{a, \{a, d\}\} = \{b, c\}$ and $|A_6| = 2$.

5. List the elements in each of the six sets P, Q, $P \cup Q$, $P \cap Q$, $P \setminus Q$ and $Q \setminus P$, where

$$P = \{x \mid x \in \mathbb{Z} \text{ and } 4 \le x \le 10\},$$
$$Q = \{y \mid y \in \mathbb{Z} \text{ and } \frac{y}{2} \in \mathbb{Z} \text{ and } 0 \le y^2 \le 50\}.$$

Solution.

The answers are:

$$P = \{4, 5, 6, 7, 8, 9, 10\}, \quad Q = \{-6, -4, -2, 0, 2, 4, 6\},$$
$$P \cup Q = \{-6, -4, -2, 0, 2, 4, 5, 6, 7, 8, 9, 10\}, \quad P \cap Q = \{4, 6\},$$
$$P \setminus Q = \{5, 7, 8, 9, 10\}, \quad Q \setminus P = \{-6, -4, -2, 0, 2\}.$$

6. Let $A = \{a, b, c, d\}$. Write down all the subsets of A. How many are there?

Solution.

The subsets of A are

$$\begin{aligned}&\emptyset,\ \{a\},\ \{b\},\ \{c\},\ \{d\},\\&\{a, b\},\ \{a, c\},\ \{a, d\},\ \{b, c\},\ \{b, d\},\ \{c, d\},\\&\{a, b, c\},\ \{a, b, d\},\ \{a, c, d\},\ \{b, c, d\},\ \{a, b, c, d\}.\end{aligned}$$

There are 16 subsets of A.

7. If A and B are subsets of a set X, prove that

$$X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B).$$

Solution.

Before we start this proof we need to know a couple of things.

- We need to know that if $x \notin A \cap B$, then either $x \notin A$ or $x \notin B$ (or both). We can take this as obvious, because if both $x \in A$ and $x \in B$, then $x \in A \cap B$.
- We also need to know that if $x \notin A$ then $x \notin A \cap B$. Again, we can take this as obvious; since $x \in A \cap B$ means both $x \in A$ and $x \in B$, so if $x \notin A$ then certainly $x \notin A \cap B$.

These two things that we are taking as obvious rest on laws of logic, which will be discussed later (in Chapter 11 of Choo and Taylor).

Proof, Part 1. Suppose that $x \in X \setminus (A \cap B)$.
Then $x \in X$ and $x \notin A \cap B$. Since $x \notin A \cap B$, $x \notin A$ or $x \notin B$.
Suppose that $x \notin A$. Since we know $x \in X$, then $x \in X \setminus A$.
Similarly, if $x \notin B$ then $x \in X \setminus B$.
Since $x \notin A$ or $x \notin B$, at least one of $x \in X \setminus A$, $x \in X \setminus B$ is true.
Thus $x \in (X \setminus A) \cup (X \setminus B)$.

This first part of the proof shows that $X \setminus (A \cap B) \subseteq (X \setminus A) \cup (X \setminus B)$.

Proof, Part 2. Suppose that $x \in (X \setminus A) \cup (X \setminus B)$.
Then $x \in X \setminus A$ or $x \in X \setminus B$.
If $x \in X \setminus A$ then $x \in X$ and $x \notin A$.
Since $x \notin A$, certainly $x \notin A \cap B$, and so $x \in X \setminus (A \cap B)$.
Similarly if $x \in X \setminus B$ then $x \in X \setminus (A \cap B)$.
So in either case $x \in X \setminus (A \cap B)$.

This second part of the proof shows that $(X \setminus A) \cup (X \setminus B) \subseteq X \setminus (A \cap B)$.

The two parts of the proof together show that $X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$. ∎

This proof is written out with a lot of English words. In advanced work, proofs do tend to be written out with quite a lot of English words; symbols like $\therefore$ are not used. However, the more advanced the work, the bigger the jumps that the reader has to fill in for himself/herself.

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Problem Set 2

1. Let $A = \{a,\, b,\, c,\, \{b\}\}$. Determine which of the following statements are true?

(i) $\{b\} \in A$; (ii) $\{b\} \subseteq A$; (iii) $\{\{b\}\} \subseteq A$; (iv) $\{a\} \in A$.

Solution.

(i) True, by definition.

(ii) True, since $b \in A$.

(iii) True, since $\{b\} \in A$.

(iv) False, since $\{a\}$ is not an element in A.

2. Let $A = \{x \mid x \in \mathbb{Z},\ -3 < x \le 2\}$ and $B = \{x^2 + 1 \mid x \in \mathbb{Z},\ -3 < x \le 2\}$.

(i) Write down the elements of A and B.

(ii) Find $A \cup B$, $A \cap B$ and $A \setminus B$. (iii) Find $|A \cup B|$, $|A \cap B|$ and $|A \setminus B|$.

Solution.

(i) $A = \{-2,\, -1,\, 0,\, 1,\, 2\}$ and $B = \{1,\, 2,\, 5\}$

(ii) $A \cup B = \{-2,\, -1,\, 0,\, 1,\, 2,\, 5\}$, $A \cap B = \{1,\, 2\}$, $A \setminus B = \{-2,\, -1,\, 0\}$.

(iii) $|A \cup B| = 6$, $|A \cap B| = 2$, and $|A \setminus B| = 3$.

3. Let $A = \{x \in \mathbb{N} \mid 1 \le x^2 \le 30\}$
and $B = \{x \in \mathbb{Z} \mid x = 2y \text{ for some } y \in \mathbb{Z} \text{ and } x^2 < 50\}$.

(i) List the elements of A and B. (ii) Find $A \cup B$, $A \cap B$, $A \setminus B$, $B \setminus A$.

(iii) Find $|A \cup B|$, $|A \cap B|$, $|A \setminus B|$, $|B \setminus A|$.

Solution.

(i) $A = \{1, 2, 3, 4, 5\}$ and $B = \{-6, -4, -2, 0, 2, 4, 6\}$.

(ii) $A \cup B = \{-6, -4, -2, 0, 1, 2, 3, 4, 5, 6\}$, $A \cap B = \{2, 4\}$, $A \setminus B = \{1, 3, 5\}$,
$B \setminus A = \{-6, -4, -2, 0, 6\}$

(iii) $|A \cup B| = 10$, $|A \cap B| = 2$, $|A \setminus B| = 3$, $|B \setminus A| = 5$.

4. (*i*) Write the following using set notation:

The set G is the set of all odd integers which are greater than 22 and are not divisible by 5.

(*ii*) With G as described in part (*i*), classify each of the following statements as true or false, giving reasons.

(a) $G \subseteq \mathbb{Z}$, (b) $\mathbb{Z} \subseteq G$, (c) $G \cap \mathbb{Z} \neq \emptyset$

(d) $\mathbb{Z} \backslash G$ is the set of all even integers less than 22 which are divisible by 5.

Solution.

(*i*) Here is one way of writing G in set notation:

$$G = \{x \mid x \in \mathbb{Z}, (x-1)/2 \in \mathbb{Z}, x > 22, x/5 \notin \mathbb{Z}\}.$$

There are various other ways.

(*ii*) (a) By definition of G, $G \subseteq \mathbb{Z}$ is true.

(b) However, $\mathbb{Z} \subseteq G$ is false, because, for example, $4 \in \mathbb{Z}$ and $4 \notin G$.

(c) True; the set $G \cap \mathbb{Z}$ is not the empty set since, for example, $23 \in G \cap \mathbb{Z}$.

(d) The statement is false, since $25 \in \mathbb{Z} \setminus G$, and 25 is not even.

5. Let $A = \{a, b, c\}$, $B = \{a, \{a\}, \{b, c\}\}$ and $C = \{\{a, b\}, b, c\}$.

(*i*) What are $|A|$, $|B|$ and $|C|$?

(*ii*) Write down $A \cup B$, $A \cup B \cup C$, $A \cap B$, $A \cap C$ and $B \cap C$.

(*iii*) Write down $A \setminus B$, $B \setminus A$, $A \setminus C$, $C \setminus A$, $B \setminus C$ and $C \setminus B$.

(*iv*) Which of the following statements are true? Give reasons!

(a) $A \subseteq B$

(b) $B = C$

(c) $\{a\} \in A$

(d) $\{a\} \in B$

(e) $\{\{a\}\} \subseteq B$

(f) $A \subseteq C$

(g) $\{a, b\} \subseteq A$

(h) $\{a, b\} \subseteq C$

(i) $\{a, b\} \in C$

(j) $\{\{a, b\}\} \subseteq C$

Solution.

(*i*) $|A| = |B| = |C| = 3$

(*ii*) We see that

$$\begin{aligned} A \cup B &= \{a, b, c, \{a\}, \{b, c\}\}, \\ A \cup B \cup C &= \{a, b, c, \{a\}, \{b, c\}, \{a, b\}\}, \\ A \cap B &= \{a\}, \\ A \cap C &= \{b, c\}, \\ B \cap C &= \emptyset. \end{aligned}$$

(*iii*) We have

$$
\begin{aligned}
A \setminus B &= \{b,\, c\},\\
B \setminus A &= \{\{a\},\, \{b,\, c\}\},\\
A \setminus C &= \{a\},\\
C \setminus A &= \{\{a,\, b\}\},\\
B \setminus C &= B,\\
C \setminus B &= C.
\end{aligned}
$$

(*iv*) (a) False: b is an element of A, but not of B.
(b) False: $a \in B$, but $a \notin C$.
(c) Clearly it is false.
(d) True: since $a \in B$.
(e) True: since $\{a\} \in B$.
(f) False: $a \in A$, but $a \notin C$.
(g) True: since both a and b are in A.
(h) False: since $a \notin C$.
(i) True: by definition.
(j) True: since $\{a,\, b\} \in C$.

6. Let A, B and C be any three sets. Prove that

$$(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)\,.$$

Solution.

Proof, Part 1. Suppose that $x \in (A \cup B) \setminus C$.
Then ($x \in A$ or $x \in B$) and $x \notin C$.
If $x \in A$ then, since $x \notin C$, we have $x \in A \setminus C$.
Similarly, if $x \in B$ then $x \in B \setminus C$.
So in either case, $x \in (A \setminus C) \cup (B \setminus C)$.

Proof, Part 2. Suppose that $x \in (A \setminus C) \cup (B \setminus C)$.
Then $x \in A \setminus C$ or $x \in B \setminus C$.
Suppose that $x \in A \setminus C$. Then $x \in A$ and $x \notin C$.
Since $x \in A$, certainly $x \in A \cup B$; since also $x \notin C$, $x \in (A \cup B) \setminus C$.
Similarly, if $x \in B \setminus C$ it follows that $x \in (A \cup B) \setminus C$.
So in either case $x \in (A \cup B) \setminus C$. ■

Here is a more compressed way of writing more or less the same proof.

We have

$$
\begin{aligned}
x \in (A \cup B) \setminus C &\Leftrightarrow \big((x \in A) \text{ or } (x \in B)\big) \;\text{ and }\; (x \notin C)\\
&\Leftrightarrow \big((x \in A) \text{ and } (x \notin C)\big) \;\text{ or }\; \big((x \in B) \text{ and } (x \notin C)\big)\\
&\Leftrightarrow \big(x \in (A \setminus C)\big) \;\text{ or }\; \big(x \in (B \setminus C)\big)\\
&\Leftrightarrow x \in (A \setminus C) \cup (B \setminus C)
\end{aligned}
$$

and so

$$(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)\,. \blacksquare$$

Either proof is acceptable; for the second one you have to be sure that all the steps really are reversible.

7. Prove that if A and B are subsets of X, then

$$X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B).$$

Solution.

Proof, Part 1. Suppose that $x \in X \setminus (A \cup B)$.
Then $x \in X$ and $x \notin A \cup B$, so $x \in X$ and $x \notin A$ and $x \notin B$.
Since $x \in X$ and $x \notin A$, $x \in X \setminus A$.
Since $x \in X$ and $x \notin B$, $x \in X \setminus B$.
Therefore $x \in (A \setminus A) \cap (X \setminus B)$.

Proof, Part 2. Suppose that $x \in (X \setminus A) \cap (X \setminus B)$.
Then $x \in X \setminus A$,so $x \in X$ and $x \notin A$.
Also $x \in X \setminus B$, so $x \in X$ (which we already knew) and $x \notin B$.
Since $x \notin A$ and $x \notin B$, $x \notin A \cup B$.
Since $x \in X$ and $x \notin A \cup B$, $x \in X \setminus (A \cup B)$. ∎

Here is a more compressed way of writing this proof.

Take any x. Then

$$\begin{aligned} x \in X \setminus (A \cup B) &\Leftrightarrow x \in X \text{ and } x \notin (A \cup B) \\ &\Leftrightarrow x \in X \text{ and } x \notin A \text{ and } x \notin B \\ &\Leftrightarrow x \in (X \setminus A) \text{ and } x \in (X \setminus B) \\ &\Leftrightarrow x \in (X \setminus A) \cap (X \setminus B). \blacksquare \end{aligned}$$

Again either proof is acceptable; again for the second one you have to be sure that all the steps really are reversible.