The University of Sydney

MATH 1004

Second Semester	Discrete Mathematics	2011

Tutorial 3 Week 4

1. Define $f : \mathbb{N} \to \mathbb{N}$ by $f(x) = x + 1$. Determine whether or not f is

(a) one-to-one; (b) onto.

Solution.

(a) If $x \neq y$, then clearly $x + 1 \neq y + 1$ so that $f(x) \neq f(y)$. Hence f is one-to-one.

(b) Since $x + 1 \geq 1$ for all $x \in \mathbb{N}$, we see that there is no $x \in \mathbb{N}$ such that $f(x) = x + 1 = 0$ and hence f is not onto.

2. Each of the following sets of pairs may or may not represent a function from $\{1, 2, 3\}$ to $\{a, b, c, d\}$.

$$\{(1, d), (2, b), (3, d)\}, \quad \{(1, c), (2, a), (3, b)\}, \quad \{(1, a), (3, b)\}$$

$$\{(1, a), (1, c), (3, d)\}, \quad \{(2, b), (3, c), (1, d)\}$$

(*i*) Identify the sets which represent functions and determine which of these are one-to-one.

(*ii*) Explain clearly why each of the sets does or does not represent a function.

(*iii*) Explain clearly why each of the sets does or does not represent a one-to-one function.

Solution.

(a) The set $\{(1, d), (2, b), (3, d)\}$ represents a function since each of the elements 1,2,3 appears exactly once as the first term of an ordered pair. It is not one-to-one since d appears more than once as the second term of an ordered pair.

(b) The set $\{(1, c), (2, a), (3, b)\}$ represents a function since each of the elements 1,2,3 appears exactly once as the first term of an ordered pair. It is one-to-one since none of the elements a,b,c,d appear more than once as second terms of ordered pairs.

(c) The set $\{(1, a), (3, b)\}$ does not represent a function on the set 1,2,3 since 2 does not appear as the first term of an ordered pair.

(d) The set $\{(1, a), (1, c), (3, d)\}$ does not represent a function since 1 appears more than once as the first term of an ordered pair. (Also, 2 does not appear as the first term of an ordered pair).

(e) The set $\{(2,b),(3,c),(1,d)\}$ represents a function since each of 1,2,3 appears exactly once as the first term of an ordered pair. It is one-to-one since none of a,b,c,d appear more than once as the second term of an ordered pair.

3. (*i*) Let $A = \{-1,2,3,5,7,11\}$ and let $B = \{1,2,\ldots,200\}$. Is the function $f : A \to B$ given by $f(x) = x^2$ one-to-one?

(*ii*) Now suppose that $A = \{-2,-1,2,3,5,7,11\}$ and $B = \{1,2,\ldots,200\}$. Is the function $f : A \to B$ given by $f(x) = x^2$ one-to-one?

Solution.

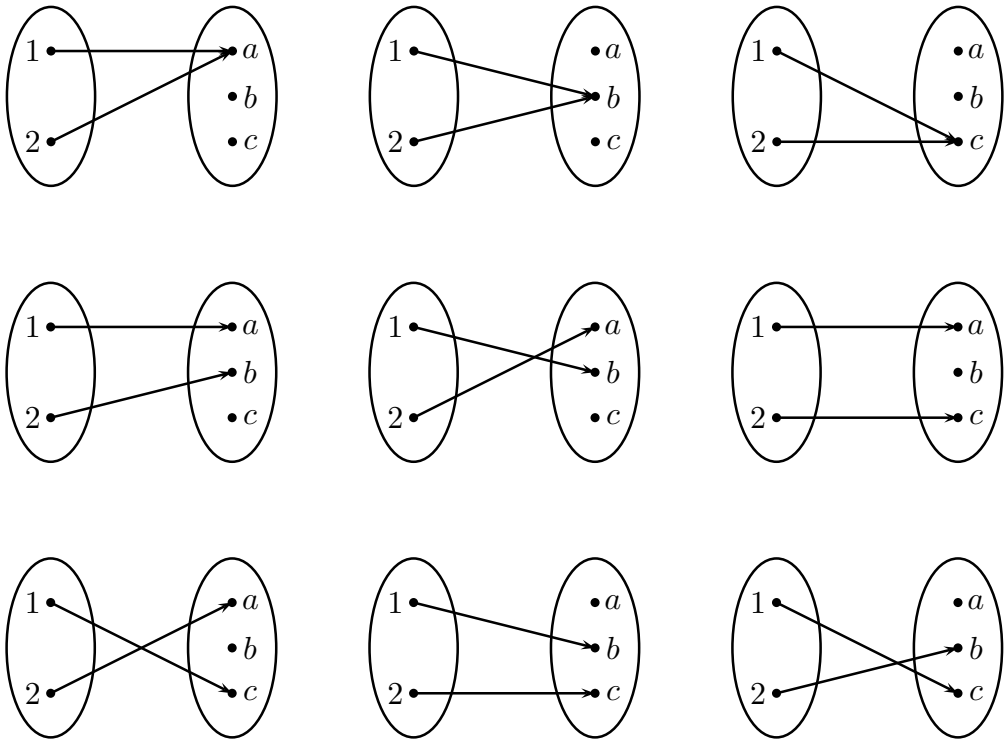
(*i*) It is easy to see that different elements in A are mapped to different elements in B, so the given function f is one-to-one.

(*ii*) Since $f(-2) = f(2) = 4$, it follows that the given function f is not one-to-one.

4. Use arrow diagrams to write down all the functions from the set $\{1,2\}$ to the set $\{a,b,c\}$. How many are there? How many one-to-one functions and how many onto functions?

Solution.

The $3^2 = 9$ functions are

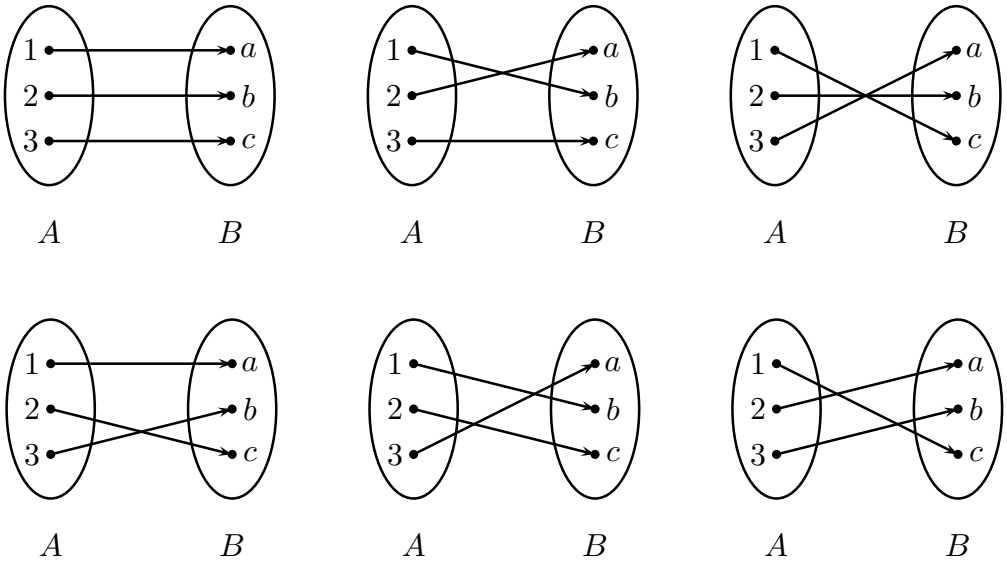


There are 6 one-to-one functions. Since there are more elements in the second set, there are no onto functions.

5. Let $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$. Write down all the one-to-one correspondences between A and B.

Solution.

The six one-to-one correspondences between A and B are



6. For each of the following functions, determine whether it is one-to-one and/or onto.

(*i*) $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = x^2$.

(*ii*) $f : \mathbb{N} \to \mathbb{Z}$ given by $f(x) = x^2$.

(*iii*) $f : \mathbb{N} \to \mathbb{N}$ given by $f(x) = x^2$.

(*iv*) $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$.

(*v*) $f : \mathbb{R}^+ \to \mathbb{R}$ given by $f(x) = x^2$.

(*vi*) $f : \mathbb{R}^+ \to \mathbb{R}^+$ given by $f(x) = x^2$.

In this question $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$.

Solution.

(*i*) f is not one-to-one since $f(1) = f(-1) = 1$. f is not onto since $f(x) = x^2$ is always positive and so for $-1 \in \mathbb{Z}$, there is no $x \in \mathbb{Z}$ with $f(x) = -1$.

(*ii*) If $f(x) = f(y)$ for $x, y \in \mathbb{N}$, then $x^2 = y^2$ and so $x = y$ since both $x, y \in \mathbb{N}$. Hence f is one-to-one. f is not onto since $f(x) = x^2$ is always positive and so for $-1 \in \mathbb{Z}$, there is no $x \in \mathbb{N}$ with $f(x) = -1$.

(*iii*) Again f is one-to-one. But f is not onto, since for $2 \in \mathbb{N}$, there is no $x \in \mathbb{N}$ such that $f(x) = x^2 = 2$.

(*iv*) f is not one-to-one since $f(1) = f(-1) = 1$. f is not onto since $f(x) = x^2$ is always positive and so for $-1 \in \mathbb{R}$, there is no $x \in \mathbb{R}$ with $f(x) = -1$.

(*v*) If $f(x) = f(y)$ for $x, y \in \mathbb{R}^+$, then $x^2 = y^2$ and so $x = y$ since both x, y are positive. Hence f is one-to-one. f is not onto since $f(x) = x^2$ is always positive and so for $-1 \in \mathbb{R}$, there is no $x \in \mathbb{R}^+$ with $f(x) = -1$.

(*vi*) Again f is one-to-one. For any $y \in \mathbb{R}^+$, we see that $x = \sqrt{y}$ is in $\mathbb{R}^+$ and that $f(x) = (\sqrt{y})^2 = y$. Hence f is onto.

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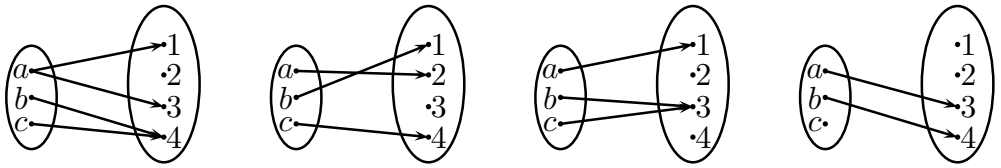
Problem Set 3

1. Let $A = \{a, b, c\}$ and $B = \{1, 2, 3, 4\}$. Which of the following arrow diagrams determine functions from A to B? Which of them determine a one-to-one functions?



(i) (ii) (iii) (iv)

Solution.

(i) The arrow diagram does not determine a function, since the element a in A is not assigned (or sent) to a unique element in B, (i.e., there are two arrows coming out of a, one pointing to 1 and the other to 3).

(ii) The arrow diagram determine a function from A to B and since distinct elements in A are mapped to distinct elements in B, the function is one-to-one.

(iii) The arrow diagram determine a function from A to B. Since both b and c are mapped to the same element 3, (i.e., the arrows coming out of b and c have the same end points 3), the function is not one-to-one.

(iv) The arrow diagram does not determine a function, since there is an element c in A which is not assigned (or sent) to any element in B, (i.e., there is no arrow coming out of c).

2. (i) Use arrow diagrams to write down all the functions from the set $\{1, 2\}$ to $\{a, b\}$.

(ii) Let $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c\}$. Let $f : A \to B$ be the function given by the 4-tuple (b, a, c, a). Draw the arrow diagram of f. Is f injective? Is f surjective?

(iii) Let $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$. Does the following set of pairs

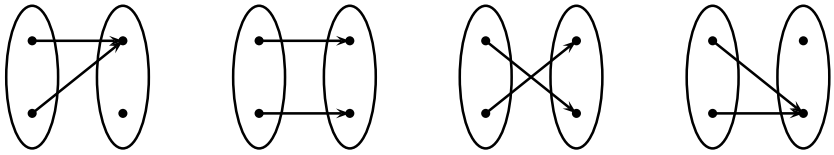
$$F = \{(a, 2), (b, 1), (c, 2)\}$$

represent a function g from A to B.

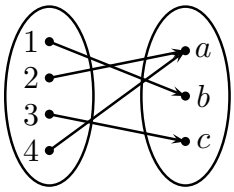
If so, is g injective? Is g surjective?

Solution.

(i) The functions are:



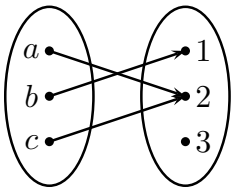
(*ii*) The arrow diagram is as shown:



Since 2 and 4 are assigned to the same element a in B, f is not injective. Since each element in B is assigned to some element in A, f is surjective.

(*iii*) Since each element in A appears once in F, F represents a function g from A to B. Since 2 appears twice in F, g is not injective. Since 3 does not appear in F, g is not surjective.

OR: The arrow diagram is as shown:



Since a and c are mapped to the same element 2 in B, g is not injective. Since there is no arrow ending at 3, g is not surjective.

3. Let $A = \{0, 1, 2, 3\}$ and $B = \{0, 1, 2, 3, 4, 5\}$, and consider the function $f : A \to B$ with rule $f(x) = x^2 - 4x + 4$ for all $x \in A$.

(*i*) Draw an arrow diagram to represent f. Write f as a set of ordered pairs of integers.

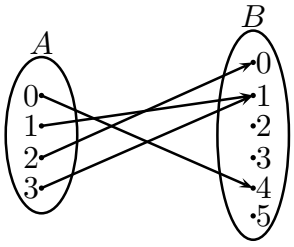
(*ii*) Is f injective? Give reasons.

(*iii*) Find the image of f. Is f surjective? Give reasons.

(*iv*) Find a set C such that with the same domain A,
$f : A \to C,\ f(x) = x^2 - 4x + 4$
is surjective.

Solution.

(*i*) We see that $f(0) = 4$, $f(1) = 1$,
$f(2) = 0$, $f(3) = 1$.
The arrow diagram is as shown:



f as a set of ordered paired of integers:

$$f = \{(0, 4), (1, 1), (2, 0), (3, 1)\}.$$

(*ii*) The function is not injective since $f(1) = f(3)$; that is, distinct elements of A do not map to distinct elements of B.

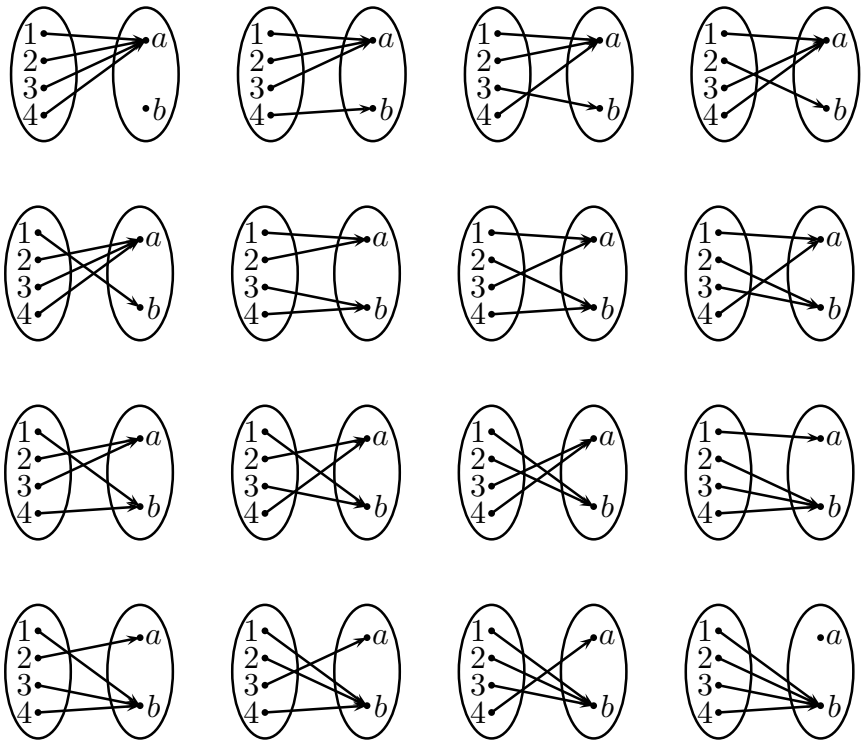
(*iii*) The image of f is $\{0, 1, 4\}$. The function is not surjective since there is no element of A which maps to 2 (or 3 or 5).

(*iv*) If $C = \{0, 1, 4\}$ then $f : A \to C$ with the rule $f(x) = x^2 - 4x + 4$ will be surjective, since $0 = f(2)$, $1 = f(1)$, $4 = f(0)$.

4. Use arrow diagrams to write down all the functions from the set $\{1, 2, 3, 4\}$ to the set $\{a, b\}$. How many are there? How many one-to-one functions and how many onto functions?

Solution.

The $2^4 = 16$ functions are:



There are 14 onto functions. Since there are more elements in the first set, there is no one-to-one function.