THE UNIVERSITY OF SYDNEY

MATH 1004

Second Semester | Discrete Mathematics | 2011

**Tutorial 4 Week 5**

**1.** (*i*) If $A = \{1, 2\}$ and $B = \{a, b, c\}$, write down the set $A \times B$.

(*ii*) For $A = \{1, 2, 3, 4\}$, write down the subset of $A \times A$ consisting of all those ordered pairs $(x, y)$ such that $x \leq y$.

(*iii*) For $A$ as above, let $D$ be the subset of $A \times A$ consisting of all ordered pairs $(x, y)$ such that $x = y$. What is $|D|$? Find a one-to-one correspondence between $D$ and $A$.

*Solution.*

(*i*) We have $A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$

(*ii*) The subset of $A \times A$ consisting of all ordered pairs $(x, y)$ such that $x \leq y$ is $\{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$.

(*iii*) We have $D = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$ and $|D| = 4$. One of the one-to-one correspondences between $D$ and $A$ is given by

$$(1, 1) \longleftrightarrow 1, \ (2, 2) \longleftrightarrow 2, \ (3, 3) \longleftrightarrow 3, \ (4, 4) \longleftrightarrow 4.$$

Note that there are 23 other one-to-one correspondences between $D$ and $A$.

**2.** (*i*) How many strings of three upper case letters are there?

(*ii*) How many strings of three upper case letters and digits are there?

(*iii*) How many strings of three upper case letters and digits are there in which the first character is a letter?

*Solution.*

(*i*) Since there are 26 upper case letters, the number of strings of three upper case letters is $26 \times 26 \times 26 = 17576$.

(*ii*) There are altogether 36 upper case letters and digits. Thus the number of strings of three upper case letters and digits is $36 \times 36 \times 36 = 46656$.

(*iii*) Since the first character of the strings is a letter, there are 26 ways of choosing the first character. There are 36 ways of choosing the second character and 36 ways of choosing the third character. Hence the number of strings of three upper case letters and digits, in which the first character is a letter, is $26 \times 36 \times 36 = 33696$.

**3.** Given an alphabet of 20 consonants and 6 vowels.

($i$) In how many ways can we select a consonant and then a vowel?

($ii$) In how many ways can we make a two-letter string consisting of one consonant and one vowel?

*Solution.*

($i$) The number of ways of choosing a consonant from 20 consonants is 20 and the number of ways of choosing a vowel from 6 vowels is 6. Hence the number of ways of selecting a consonant and then a vowel is $20 \times 6 = 120$.

($ii$) The number of ways of making a two-letter word consisting of one consonant and one vowel is $20 \times 6 + 6 \times 20 = 240$.

**4.** In a town of 18,000 people everyone has three initials. Must there be two people with the same initials?

*Solution.*

The number of ways of getting three initials is $26 \times 26 \times 26 = 17576$. Since there are 18,000 people, there must be two people with the same initials.

**5.** ($i$) How many strings of four digits are there if 0 is never used?

($ii$) How many six digit numbers are there which do not repeat a digit and do not begin with 0?

($iii$) How many strings of length 3 start with 2 digits and end with one of the 26 capital letters of the alphabet?

*Solution.*

($i$) Since 0 is never used, we see that there are 9 ways to choose each digit. Hence the number of strings of four digits, if 0 is never used, is $9^4 = 6561$.

($ii$) Since the digits do not begin with 0, there are 9 ways of choosing the first digit of the six digit numbers. Since the numbers do not repeat a digit, there are 9 ways of choosing the second digit and then 8 ways of choosing the third, 7 ways the fourth, 6 ways the fifth and 5 ways the last digit. Hence the total number of six digit numbers, which do not repeat a digit and do not begin with 0, is $9 \times 9 \times 8 \times 7 \times 6 \times 5 = 136080$.

($iii$) There are 10 ways to choose the first digit and second digit, and there are 26 ways of choosing the capital letter. Hence the number of strings of length 3 start with 2 digits and one capital letter is $10 \times 10 \times 26 = 2600$.

**6.** (*i*) How many four digit numbers greater than 1000 can be formed using the digits 0, 1, 2, 3 and 4?

(*ii*) How many four digit numbers greater than 1000, with no repeated digit, can be formed using the digits 0, 1, 2, 3 and 4?

*Solution.*

(*i*) Since the number is greater than 1000, there are 4 ways to choose the first digit. Since the digits can be repeated, there are 5 ways to choose the second digit, the third digit and the fourth digit. Thus the number of four digit numbers greater than or equal to 1000 is $4 \times 5 \times 5 \times 5 = 500$. Hence the number of four digit numbers greater than 1000 is $500 - 1 = 499$.

(*ii*) Since the number is greater than 1000, there are 4 ways to choose the first digit. Since the digits cannot be repeated, there are 4 ways to choose for the second digit, then 3 ways for the third digit and 2 ways for the fourth digit. Hence the number of four digit numbers greater than 1000, with no repeated digit, is $4 \times 4 \times 3 \times 2 = 96$.

**7.** Four people are about to have a snack and there are eleven types of cake available. Each person chooses just one cake.

(*i*) How many possibilities are there?

(*ii*) How many possibilities are there if everyone has a different type of cake?

*Solution.*

(*i*) Each person has 11 choices so that the number of possible ways is $11^4$.

(*ii*) If everyone has a different type of cake, then the number of possible ways is $11_{(4)} = 11 \times 10 \times 9 \times 8 = 7920$.

## Problem Set 4

**1.** Suppose that there are 8 different kinds of doughnuts in a coffee shop.

($i$) In how many ways for 5 students, to each buy 1 doughnut?

($ii$) In how many ways for 5 students, to each buy 1 doughnut so that no two students buy the same kind?

*Solution.*

($i$) Each student has 8 ways to buy a doughnut and so the number of ways for 5 students, to each buy 1 doughnut is $8^5 = 32768$.

($ii$) The first student has 8 ways to buy a doughnut. Since no two students buy the same doughnut, the second student has only 7 ways to buy a doughnut and so on. Thus the number of ways for 5 students, to each buy 1 doughnut so that no two students buy the same kind, is
$8_{(5)} = 8 \times 7 \times 6 \times 5 \times 4 = 6720$.

**2.** ($i$) How many 5-digit numbers, greater than 60000, can be formed from the digits 2, 3, 4, 5, 6, 7, 8, 9?

($ii$) How many 5-digit numbers, greater than 60000, can be formed from the digits 3, 4, 5, 6, 7, 8, 9, 0?

*Solution.*

($i$) There are 4 ways to choose for the first digit and 8 ways to choose for each of the remaining digit and so the number of ways of forming such 5-digit numbers is $4 \times 8^4$.

($ii$) There are 4 ways to choose for the first digit and 8 ways to choose for each of the remaining digit and so the number of ways of forming such 5-digit numbers (including the number 60000) is $4 \times 8^4$. Hence there are $4 \times 8^4 - 1$ 5-digit numbers, greater than 60000, can be formed from the given digits.

**3.** (*i*) In how many ways can we list 4 novels followed by 3 biographies if there are 8 novels and 6 biographies from which to choose?

(*ii*) How many 4-digit numbers greater than 7000 can be formed from the digits 1 to 9?

*Solution.*

(*i*) There are $8_{(4)}$ ways to list the four novels and $6_{(3)}$ ways to list the six biographies. Hence the number of ways of listing four novels followed by six biographies is $8_{(4)} \times 6_{(3)}$ or $8 \times 7 \times 6 \times 5 \times 6 \times 5 \times 4 = 201600$.

(*ii*) Since the number is greater than 7000, there are 3 ways to choose for the first digit. There are 9 ways to choose for the second, the third and the fourth digit. Hence there are $3 \times 9^3$ 4-digit numbers greater than 7000.

**4.** (*i*) How many strings of four digits are there (with repetition) if the first digit is not 0?

(*ii*) How many strings of four digits are there without repetition if the first digit is not 0?

(*iii*) In how many ways can 8 children be seated on a bench?

(*iv*) In how many ways can 8 children be seated on a bench if three particular children must sit together?

(*v*) In how many ways can 8 children be seated on a bench if three particular children must **not** sit together?

*Solution.*

(*i*) Since the first digit is not 0, there are 9 ways to choose the first digit. There are 10 ways to choose for the second, third and fourth digit. Hence the number of strings of four digits, if the first digit is not 0, is $9 \times 10 \times 10 \times 10 = 9000$.

(*ii*) Again there are 9 ways to choose the first digit. since the strings do not repeat a digit, there are 9 ways to choose the second digit and then 8 ways to choose the thid and 7 ways to choose the fourth. Hence the number of strings of four digits without repetition, if the first digit is not 0, is $9 \times 9_{(3)} = 9 \times 9 \times 8 \times 7 = 4536$.

(*iii*) Since there are 8 children, there are $8! = 40320$ ways the children can be seated on a bench.

(*iv*) Think of the three particular children as one unit, and then there are 6 units to be seated. This can be done in 6! ways. But the three children can be arranged in 3! ways in each of these 6! ways. Hence the answer is $3! \times 6! = 4320$.

(*v*) There are 8! ways of arranging the children without restriction, and there are $3! \times 6!$ arrangements in which the three children are together (see (*iv*) and (*v*)). Thus the required number is $8! - 3! \times 6! = 50 \times 6! = 36000$.

**5.** (*i*) A restaurant has five entrées, seven main courses and ten desserts. In how many ways can you select two dishes on the condition that they must not both be from the same part of the menu?

(*ii*) A dictionary consists of an alphabetical arrangement of all possible "words" of 3 or fewer letters, using the 26 letters of the alphabet. (For example, b,bz, cat, dog, xta ) The dictionary is in two volumes, A-M and N-Z. How many words are there in the A-M volume?

(*iii*) In how many ways can 8 children be seated on a bench if two particular children must sit together?

(*iv*) In how many ways can 8 children be seated on a bench if two particular children must **not** sit together?

*Solution.*

(*i*) The two dishes can be selected as follows: entrée, main course; entrée, dessert; or main course, dessert.

There are 5 ways of choosing entrées, 7 ways of choosing main courses and 10 ways of choosing desserts. Then there are $5 \times 7$ ways of choosing an entrée and main course; $5 \times 10$ ways of choosing an entrée and dessert; and $7 \times 10$ ways of choosing the a main course and dessert.

Hence the number of ways of selecting two dishes, on the condition that they must not both be from the same part of the menu, is $35 + 50 + 70 = 155$.

(*ii*) There are 13 words of one letter in the A-M volume. Words of two letters in this volume can be formed in $13 \times 26$ ways and words of three letters can be formed in $13 \times 26^2$ ways. The total number of words is therefore $13(1 + 26 + 26^2) = 9139$.

(*iii*) Think of the two particular children as one unit, and then there are 7 units to be seated. This can be done in 7! ways. The two children can be arranged in 2 ways in each of these 7! ways, so the answer is $2 \times 7! = 10080$.

(*iv*) There are 8! ways of arranging the children without restriction, and there are $2 \times 7!$ arrangements in which the two children are together (see (*iii*)). Therefore the required number is $8! - 2 \times 7! = 6 \times 7! = 30240$.