The University of Sydney

MATH 1004

Second Semester	Discrete Mathematics	2011

Tutorial 9 Week 10

1. Draw truth tables for the following propositions.

(i) $\sim(p \vee q)$.

(ii) $\sim p \wedge \sim q$.

(iii) $p \wedge (p \Rightarrow q)$.

Solution.

The truth tables for the given propositions are shown as follows:

(i)

p	q	$p \vee q$	$\sim(p \vee q)$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

(ii)

p	q	$\sim p$	$\sim q$	$\sim p \wedge \sim q$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

(iii)

p	q	$p \Rightarrow q$	$p \wedge (p \Rightarrow q)$
T	T	T	T
T	F	F	F
F	T	T	F
F	F	T	F

2. Show that the following pairs of propositions are equivalent:

(i) $(p \vee q) \Rightarrow r$; $(\sim p \wedge \sim q) \vee r$.

(ii) $p \Rightarrow \sim q$; $q \Rightarrow \sim p$.

Solution.

In each case, we need to show that the two given propositions have the same truth table.

(i) The truth tables for the two given propositions are:

p	q	r	$p \vee q$	$(p \vee q) \Rightarrow r$
T	T	T	T	T
T	T	F	T	F
T	F	T	T	T
T	F	F	T	F
F	T	T	T	T
F	T	F	T	F
F	F	T	F	T
F	F	F	F	T

p	q	r	$\sim p$	$\sim q$	$\sim p \wedge \sim q$	$(\sim p \wedge \sim q) \vee r$
T	T	T	F	F	F	T
T	T	F	F	F	F	F
T	F	T	F	T	F	T
T	F	F	F	T	F	F
F	T	T	T	F	F	T
F	T	F	T	F	F	F
F	F	T	T	T	T	T
F	F	F	T	T	T	T

Since the given propositions have the same truth tables, they are equivalent.

(ii) The truth tables for the given propositions are:

p	q	$\sim q$	$p \Rightarrow \sim q$
T	T	F	F
T	F	T	T
F	T	F	T
F	F	T	T

p	q	$\sim p$	$q \Rightarrow \sim p$
T	T	F	F
T	F	F	T
F	T	T	T
F	F	T	T

Since the given propositions have the same truth tables, they are equivalent.

3. Decide whether the following propositions are true or false:

(i) If $1+1=2$, then $2+2=5$.

(ii) If $1+1\neq 2$, then pigs might fly.

(iii) If $1+1=2$, then $2+3=5$.

Solution.

Each of the given propositions is of the form $p \Rightarrow q$.

(i) In this case, p is true and q is false and so the given proposition is false.

(ii) In this case, p is false and so the given proposition is true, whether q is true or false.

(iii) In this case, p is true and q is true and so the given proposition is true.

4. Show that $\sim(p \vee q) \vee (\sim p \wedge q) \vee p$ is a tautology.

Solution.

Let $s = \sim(p \vee q) \vee (\sim p \wedge q) \vee p$. Then

p	q	$\sim(p \vee q)$	$\sim p$	$\sim p \wedge q$	$\sim(p \vee q) \vee (\sim p \wedge q)$	s
T	T	F	F	F	F	T
T	F	F	F	F	F	T
F	T	F	T	T	T	T
F	F	T	T	F	T	T

Since every entry in the last column is T, the given proposition is a tautology.

5. Show that $(p \wedge q) \wedge \sim(p \vee q)$ is a contradiction.

Solution.

We have

p	q	$p \vee q$	$\sim(p \vee q)$	$(p \wedge q)$	$(p \wedge q) \wedge \sim(p \vee q)$
T	T	T	F	T	F
T	F	T	F	F	F
F	T	T	F	F	F
F	F	F	T	F	F

Since every entry in the last column is F, the given proposition is a contradiction.

6. Taking the universal set to be the set $\mathbb{R}$ of all real numbers, determine the truth or falsity of the following sentences.

(i) $(\forall x)\big((x \in \mathbb{Z}) \Rightarrow (x^2 - x - 1 > 0)\big)$.

(ii) $(\exists x)\big((x \in \mathbb{Z}) \wedge (x^2 - x - 1 > 0)\big)$.

(iii) $(\forall x)\big((x^2 = 1) \Rightarrow (x = 1)\big)$.

(iv) $(\exists x)\big((x^2 = 1) \wedge (x = 1)\big)$.

Solution.

(i) The sentence is false, by taking $x = 0$.

(ii) The sentence is true, by taking $x = 2$.

(iii) The sentence is false, by taking $x = -1$.

(iv) The sentence is true, by taking $x = 1$.

7. Write the following propositions in symbolic form:

(i) The square of every real number is non negative.

(ii) There is an x in the set A which is not in the set B.

Solution.

(i) There are generally many ways of writing a given proposition in symbolic form. Three appropriate answers would be:

$$(\forall x)((x \in \mathbb{R}) \Rightarrow (x^2 \notin (-\infty, 0)))$$

or

$$(\forall x)((x \in \mathbb{R}) \Rightarrow (x^2 \geq 0)))$$

or

$$(\forall x)((x \in \mathbb{R}) \Rightarrow \sim(x^2 \in (-\infty, 0))).$$

(ii) The answers would be:

$$(\exists x)((x \in A) \wedge (x \notin B))$$

or

$$(\exists x)((x \in A) \wedge \sim(x \in B)).$$

8. Write the following propositions in symbolic form:

(i) All hungry crocodiles are not amiable.

(ii) Some crocodiles, if not hungry, are amiable.

Solution.

Suppose that the universal set is the set of all crocodiles x. Then suppose

$H(x)$ means "x is hungry."
$M(x)$ means "x is amiable."

Note that there are different ways of writing the propositions in symbolic form. We will present one of them.

(i) The proposition "all hungry crocodiles are not amiable" can be written in symbolic form as:

$$(\forall x)\big(H(x) \Rightarrow \sim M(x)\big).$$

(ii) The proposition "some crocodiles, when not hungry, are amiable" can be written in symbolic form as:

$$(\exists x)\big(\sim H(x) \Rightarrow M(x)\big).$$

Problem Set 9

1. (*i*) Show that $((p \vee q) \wedge \sim q) \Rightarrow p$ is a tautology.

(*ii*) Show that $(p \vee q) \wedge (\sim p \wedge \sim q)$ is a contradiction.

Solution.

(*i*) We have

p	q	$(p \vee q)$	$\sim q$	$((p \vee q) \wedge \sim q)$	$((p \vee q) \wedge \sim q) \Rightarrow p$
T	T	T	F	F	T
T	F	T	T	T	T
F	T	T	F	F	T
F	F	F	T	F	T

Since every entry in the last column is T, the given proposition is a tautology.

(*ii*) We have

p	q	$(p \vee q)$	$\sim p$	$\sim q$	$(\sim p \wedge \sim q)$	$(p \vee q) \wedge (\sim p \wedge \sim q)$
T	T	T	F	F	F	F
T	F	T	F	T	F	F
F	T	T	T	F	F	F
F	F	F	T	T	T	F

Since every entry in the last column is F, the given proposition is a contradiction.

2. For each of the following propositions, write down its truth table and determine whether or not it is a tautology or contradiction.

(*i*) $(p \wedge \sim q) \Rightarrow \sim (p \Rightarrow q)$

(*ii*) $(p \wedge \sim q) \wedge (\sim p \vee q)$

Solution.

(*i*) Let $s = (p \wedge \sim q) \Rightarrow \sim (p \Rightarrow q)$. Then the truth table for the proposition s is

p	q	$\sim q$	$p \wedge \sim q$	$(p \Rightarrow q)$	$\sim (p \Rightarrow q)$	s
T	T	F	F	T	F	T
T	F	T	T	F	T	T
F	T	F	F	T	F	T
F	F	T	F	T	F	T

Since every entry in the last column is T, the given proposition is a tautology.

(*ii*) We have

p	q	$\sim p$	$\sim q$	$(p \wedge \sim q)$	$(\sim p \vee q)$	$(p \wedge \sim q) \wedge (\sim p \vee \sim q)$
T	T	F	F	F	T	F
T	F	F	T	T	F	F
F	T	T	F	F	T	F
F	F	T	T	F	T	F

Since every entry in the last column is F, the given proposition is a contradiction.

3. (*i*) Draw truth table for the proposition

$$(p \wedge q) \vee \sim(p \Rightarrow q),$$

and determine whether it is a tautology or a contradiction or neither.

(*ii*) Taking the universal set to be the set $\mathbb{R}$ of all real numbers, determine the truth and falsity of the following propositions.

(a) $(\forall x)((x > 2) \Rightarrow (x^2 > 4))$

(b) $(\forall x)((x^2 > 4) \Rightarrow (x > 2))$

(c) $(\exists x)((x > 2) \Rightarrow (x^2 > 4))$

(d) $(\exists x)((x^2 > 4) \Rightarrow (x > 2))$

Solution.

(*i*) The truth table for the given proposition is

p	q	$(p \wedge q)$	$(p \Rightarrow q)$	$\sim(p \Rightarrow q)$	$(p \wedge q) \vee \sim(p \Rightarrow q)$
T	T	T	T	F	T
T	F	F	F	T	T
F	T	F	T	F	F
F	F	F	T	F	F

Since not every entry in the last column is true, it is not a tautology; and since not every entry in the last column is false, it is not a contradiction.

(*ii*) (a) The sentence is clearly true.

(b) The sentence is false, by taking $x = -3$. Then $x^2 = 9 > 4$ is true, but $x > 2$ is false.

(c) The sentence is true, by taking $x = 3$.

(d) The sentence is true, by taking $x = 3$.

4. (*i*) Draw a truth table for each of the following propositions, and determine whether it is a tautology or a contradiction or neither.

(a) $(p \Rightarrow q) \Rightarrow (\sim p \vee q)$,

(b) $(p \wedge q) \vee \sim(p \Rightarrow q)$.

(*ii*) Let $\mathbb{Z}$ be the set of all integers, $\mathbb{N}$ the set of all natural numbers and $\mathbb{R}$ the set of all real numbers. Determine the truth and falsity of the following propositions.

(a) $(\forall x \in \mathbb{Z})((x^2 = 9) \Rightarrow (x = 3))$

(b) $(\forall x \in \mathbb{N})((x^2 = 9) \Rightarrow (x = 3))$

(c) $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})((x > 0) \Rightarrow (x = y^2))$

(d) $(\forall x \in \mathbb{Z})(\exists y \in \mathbb{R})((x > 0) \Rightarrow (x = y^2))$

Solution.

(*i*) (a) The truth table for the given proposition is

p	q	$(p \Rightarrow q)$	$\sim p$	$(\sim p \vee q)$	$(p \Rightarrow q) \Rightarrow (\sim p \vee q)$
T	T	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

Since every entry in the last column is T, the proposition is a tautology.

(b) The truth table for the given proposition is

p	q	$(p \wedge q)$	$(p \Rightarrow q)$	$\sim(p \Rightarrow q)$	$(p \wedge q) \vee \sim(p \Rightarrow q)$
T	T	T	T	F	T
T	F	F	F	T	T
F	T	F	T	F	F
F	F	F	T	F	F

Since not every entry in the last column is true, it is not a tautology; and since not every entry in the last column is false, it is not a contradiction.

(*ii*) (a) If $x \in \mathbb{Z}$ and $x^2 = 9$, then $x = 3$ or -3 so that the proposition is false.

(b) If $x \in \mathbb{N}$, then $x \geq 0$ and so if $x^2 = 9$, then $x = 3$. Hence the proposition is true.

(c) The proposition is true. Suppose that $x > 0$. Since $x \in \mathbb{R}$ we see that $y = \sqrt{x} \in \mathbb{R}$ such that $x = y^2$.

(d) The proposition is again true, since all integers are real. So if $x > 0$, then we take $y = \sqrt{x}$.