

Tutorial 11 Week 12

1. Find a closed form for the generating function of the sequence a_0, a_1, \dots , where $a_0 = 1, a_1 = 1$ and

$$a_n = 5a_{n-1} - 6a_{n-2}.$$

Solution.

Let

$$G(z) = a_0 + a_1z + a_2z^2 + a_3z^3 + \dots$$

be the generating function of the given sequence a_0, a_1, a_2, \dots . After multiplying by z and z^2 we find that

$$zG(z) = a_0z + a_1z^2 + a_2z^3 + \dots$$

$$z^2G(z) = a_0z^2 + a_1z^3 + \dots$$

Then using the recurrence relation $a_n - 5a_{n-1} + 6a_{n-2} = 0$, we obtain

$$(1 - 5z + 6z^2)G(z) = a_0 + (a_1 - 5a_0)z.$$

Using the initial values $a_0 = 1$ and $a_1 = 1$, we then have

$$(1 - 5z + 6z^2)G(z) = 1 - 4z.$$

Hence the closed form of the generating function is

$$G(z) = \frac{1 - 4z}{1 - 5z + 6z^2}.$$

2. Let a_0, a_1, a_2, \dots , be a sequence of integers satisfying the recurrence relation

$$a_n - 2a_{n-1} - a_{n-2} + 2a_{n-3} = 0,$$

for $n \geq 4$, where $a_0 = 0, a_1 = 0, a_2 = 2$ and $a_3 = 5$. Find a closed form for the generating function of this sequence.

Solution.

Let

$$G(z) = a_0 + a_1z + a_2z^2 + a_3z^3 + a_4z^4 \dots$$

be the generating function of the given sequence a_0, a_1, a_2, \dots . After multiplying by z, z^2 and z^3 , we find that

$$\begin{aligned} zG(z) &= a_0z + a_1z^2 + a_2z^3 + a_3z^4 + \dots \\ z^2G(z) &= a_0z^2 + a_1z^3 + a_2z^4 + \dots \\ z^3G(z) &= a_0z^3 + a_1z^4 + \dots \end{aligned}$$

Then using the recurrence relation for $n \geq 4$, we have

$$\begin{aligned} &(1 - 2z - z^2 + 2z^3)G(z) \\ &= a_0 + (a_1 - 2a_0)z + (a_2 - 2a_1 - a_0)z^2 + (a_3 - 2a_2 - a_1 + 2a_0)z^3. \end{aligned}$$

Using the initial values $a_0 = 0, a_1 = 0, a_2 = 2$ and $a_3 = 5$, we see that

$$(1 - 2z - z^2 + 2z^3)G(z) = 2z^2 + z^3,$$

and hence the generating function in closed form is

$$G(z) = \frac{2z^2 + z^3}{1 - 2z - z^2 + 2z^3}.$$

3. Solve the following recurrence relations using generating functions:

(i) $x_n = 4x_{n-1} - 3x_{n-2}$, where $x_0 = 1$ and $x_1 = 2$.

(ii) $x_n = 3x_{n-1} - 3x_{n-2} + x_{n-3}$, where $x_0 = 0, x_1 = 1$ and $x_2 = 3$.

Solution.

(i) Let

$$G(z) = x_0 + x_1z + x_2z^2 + x_3z^3 + \dots$$

be the generating function of the given sequence x_0, x_1, x_2, \dots . After multiplying by z and z^2 , we find that

$$\begin{aligned} zG(z) &= x_0z + x_1z^2 + x_2z^3 + \dots \\ z^2G(z) &= x_0z^2 + x_1z^3 + \dots \end{aligned}$$

Then we use the recurrence relation and the initial values to get

$$(1 - 4z + 3z^2)G(z) = x_0 + (x_1 - 4x_0)z = 1 - 2z,$$

so that

$$G(z) = \frac{1 - 2z}{(1 - 4z + 3z^2)}.$$

Now we have $1 - 4z + 3z^2 = (1 - z)(1 - 3z)$ and we write

$$\frac{1 - 2z}{(1 - 4z + 3z^2)} = \frac{A}{1 - z} + \frac{B}{1 - 3z}.$$

Solving this for A and B we find that $A = B = \frac{1}{2}$. Thus

$$G(z) = \frac{1}{2} \frac{1}{1-z} + \frac{1}{2} \frac{1}{1-3z}.$$

Using the formula for $(1-z)^{-1}$, replacing z by $3z$ for the second term, we obtain

$$G(z) = \frac{1}{2}(1+z+z^2+\dots) + \frac{1}{2}(1+3z+3^2z^2+\dots).$$

Hence the solution to the recurrence relation is $x_n = \frac{1}{2} + \frac{1}{2} \cdot 3^n$.

(ii) Let

$$G(z) = x_0 + x_1z + x_2z^2 + x_3z^3 + \dots$$

be the generating function of the given sequence x_0, x_1, x_2, \dots . After multiplying by z, z^2 and z^3 , we find that

$$\begin{aligned} zG(z) &= x_0z + x_1z^2 + x_2z^3 + \dots \\ z^2G(z) &= x_0z^2 + x_1z^3 + \dots \\ z^3G(z) &= x_0z^3 + \dots \end{aligned}$$

Then we use the recurrence relation and the initial values to get

$$(1-3z+3z^2-z^3)G(z) = x_0 + (x_1-3x_0)z + (x_2-3x_1+3x_0)z^2 = z,$$

and therefore $G(z) = \frac{z}{(1-z)^3}$. But in general

$$\frac{1}{(1-x)^k} = (1-x)^{-k} = \sum_{n \geq 0} \binom{n+k-1}{n} x^n.$$

Thus

$$\begin{aligned} G(z) &= z \sum_{n \geq 0} \binom{n+2}{n} z^n = \sum_{n \geq 0} \binom{n+2}{2} z^{n+1} \\ &= \sum_{n \geq 1} \binom{n+1}{2} z^n = \sum_{n \geq 1} \frac{1}{2} n(n+1) z^n. \end{aligned}$$

Hence the solution to the recurrence relation is $x_n = \frac{1}{2}n(n+1)$.

4. Consider the recurrence relation

$$a_n = a_{n-1} + 4,$$

for $n > 0$ and where $a_0 = 1$. Find a closed form for its generating function and thereby find a formula for a_n .

Solution.

Let

$$G(z) = a_0 + a_1z + a_2z^2 + a_3z^3 + \dots$$

be the generating function of the given sequence a_0, a_1, a_2, \dots . After multiplying by z , we find that

$$zG(z) = a_0z + a_1z^2 + a_2z^3 + \dots$$

Then using the recurrence relation and the initial value for a_0 , we see that

$$\begin{aligned} (1-z)G(z) &= a_0 + (a_1 - a_0)z + (a_2 - a_1)z^2 + (a_3 - a_2)z^3 + \dots \\ &= 1 + 4z + 4z^2 + 4z^3 + \dots \\ &= 1 + 4z(1 + z + z^2 + z^3 + \dots) = 1 + \frac{4z}{1-z}. \end{aligned}$$

Hence

$$G(z) = \frac{1}{1-z} + \frac{4z}{(1-z)^2}.$$

Now

$$\begin{aligned} G(z) &= \frac{1}{1-z} + \frac{4z}{(1-z)^2} \\ &= \sum_{n=0}^{\infty} z^n + 4z \sum_{n=0}^{\infty} (n+1)z^n \\ &= \sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} 4(n+1)z^{n+1} \\ &= \sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} 4nz^n \end{aligned}$$

so that a formula for a_n is

$$a_n = 1 + 4n.$$

Problem Set 11

1. For each of the following recurrence relations, find a closed form for its generating function:

(i) $x_{n+2} - 5x_{n+1} + 6x_n = 0$ for all $n \geq 0$, with $x_0 = 1$ and $x_1 = 3$.

(ii) $a_n - 2a_{n-1} = 4^{n-1}$ for all $n \geq 1$, with $a_0 = 1$.

Solution.

(i) Let

$$G(z) = x_0 + x_1z + x_2z^2 + x_3z^3 + \dots$$

be the generating function of the given sequence x_0, x_1, x_2, \dots . After multiplying by z and z^2 , we find that

$$\begin{aligned} zG(z) &= x_0z + x_1z^2 + x_2z^3 + \dots \\ z^2G(z) &= x_0z^2 + x_1z^3 + \dots \end{aligned}$$

Then we use the recurrence relation and the initial values to get

$$(1 - 5z + 6z^2)G(z) = x_0 + (x_1 - 5x_0)z = 1 - 2z$$

so that a closed form is

$$G(z) = \frac{1 - 2z}{1 - 5z + 6z^2}.$$

(ii) Let

$$G(z) = a_0 + a_1z + a_2z^2 + a_3z^3 + a_4z^4 + \dots$$

be the generating function of the given sequence a_0, a_1, a_2, \dots . Then

$$zG(z) = a_0z + a_1z^2 + a_2z^3 + a_3z^4 + \dots$$

Then using the recurrence relation and the initial condition, we obtain

$$\begin{aligned} (1 - 2z)G(z) &= 1 + 4^0z + 4^1z^2 + 4^2z^3 + \dots \\ &= 1 + z(1 + 4z + (4z)^2 + \dots) \\ &= 1 + \frac{z}{1 - 4z} = \frac{1 - 3z}{1 - 4z}. \end{aligned}$$

Thus the closed form of the generating function is

$$G(z) = \frac{1 - 3z}{(1 - 2z)(1 - 4z)}.$$

2. Consider the following recurrence relation

$$x_n = -x_{n-1} + 6x_{n-2},$$

for $n \geq 2$, where $x_0 = 1$ and $x_1 = 1$. Find a closed form for its generating function and then find a formula for x_n .

Solution.

Let

$$G(z) = x_0 + x_1z + x_2z^2 + x_3z^3 + \dots$$

be the generating function of the given sequence x_0, x_1, x_2, \dots . After multiplying by z and z^2 , we find that

$$\begin{aligned} zG(z) &= x_0z + x_1z^2 + x_2z^3 + \dots \\ z^2G(z) &= + x_0z^2 + x_1z^3 + \dots \end{aligned}$$

Then we use the recurrence relation and the initial values to get

$$(1 + z - 6z^2)G(z) = x_0 + (x_1 + x_0)z = 1 + 2z,$$

so that the generating function in its closed form is

$$G(z) = \frac{1 + 2z}{(1 + z - 6z^2)}.$$

We have $1 + z - 6z^2 = (1 + 3z)(1 - 2z)$ and we want to find numbers A and B such that

$$\frac{1 + 2z}{(1 + 3z)(1 - 2z)} = \frac{A}{1 + 3z} + \frac{B}{1 - 2z}.$$

This gives

$$\frac{1 + 2z}{(1 + 3z)(1 - 2z)} = \frac{A(1 - 2z) + B(1 + 3z)}{(1 + 3z)(1 - 2z)}$$

and so $A + B = 1$ and $-2A + 3B = 2$. Solving these equations simultaneously yields $A = \frac{1}{5}$ and $B = \frac{4}{5}$. Thus

$$G(z) = \frac{1}{5} \frac{1}{1 + 3z} + \frac{4}{5} \frac{1}{1 - 2z}.$$

Using the formula for $(1 - z)^{-1}$, replacing z by $-3z$ for the first term and z by $2z$ in the second term, we obtain

$$G(z) = \frac{1}{5}(1 + (-3)z + (-3)^2z^2 + \dots) + \frac{4}{5}(1 + 2z + 2^2z^2 + \dots).$$

The coefficient for z^n is x_n . Hence the solution to the recurrence relation is

$$x_n = \frac{1}{5}(-3)^n + \frac{4}{5} \cdot 2^n.$$

3. Consider the following recurrence relation

$$a_{n+2} = a_{n+1} + a_n + n$$

for $n \geq 0$, where $a_0 = a_1 = 1$. Write the corresponding generating function in its closed form.

Solution.

Let

$$G(z) = a_0 + a_1z + a_2z^2 + a_3z^3 + a_4z^4 + \dots$$

be the generating function of the given sequence a_0, a_1, a_2, \dots . After multiplying by z and z^2 we find that

$$\begin{aligned} zG(z) &= a_0z + a_1z^2 + a_2z^3 + a_3z^4 + \dots \\ z^2G(z) &= a_0z^2 + a_1z^3 + a_2z^4 + \dots \end{aligned}$$

Using the recurrence relations and the initial conditions, we then obtain

$$\begin{aligned} (1 - z - z^2)G(z) &= a_0 + (a_1 - a_0)z + (a_2 - a_1 - a_0)z^2 + (a_3 - a_2 - a_1)z^3 \\ &\quad + (a_4 - a_3 - a_2)z^4 + \dots \\ &= 1 + z^3 + 2z^4 + \dots \\ &= 1 + z^3(1 + 2z + 3z^2 + \dots) = 1 + \frac{z^3}{(1 - z)^2}. \end{aligned}$$

Hence $G(z)$ in its closed form is given by

$$G(z) = \frac{1}{1 - z - z^2} + \frac{z^3}{(1 - z)^2(1 - z - z^2)}.$$

4. (i) Find a closed form for the generating function of the recurrence relation

$$x_n - 5x_{n-1} + 6x_{n-2} = 0, \quad n \geq 2, \quad x_0 = 3, \quad x_1 = 7.$$

- (ii) Find a closed form for the generating function of the recurrence relation

$$x_n = 2x_{n-1} - x_{n-2} + n, \quad n \geq 2, \quad x_0 = 1, \quad x_1 = 2$$

and then find a formula for x_n .

Solution.

- (i) Let

$$G(z) = x_0 + x_1z + x_2z^2 + x_3z^3 + \dots$$

be the generating function of the given sequence x_0, x_1, x_2, \dots . After multiplying by z and z^2 , we find that

$$\begin{aligned} zG(z) &= x_0z + x_1z^2 + x_2z^3 + \dots \\ z^2G(z) &= x_0z^2 + x_1z^3 + \dots \end{aligned}$$

Then we use the recurrence relation and the initial values to get

$$(1 - 5z + 6z^2)G(z) = x_0 + (x_1 - 5x_0)z = 3 - 8z$$

so that a closed form is

$$G(z) = \frac{3 - 8z}{1 - 5z + 6z^2}.$$

- (ii) Write the recurrence relation as:

$$x_n - 2x_{n-1} + x_{n-2} = n.$$

Let

$$G(z) = x_0 + x_1z + x_2z^2 + x_3z^3 + \dots$$

be the generating function of the given sequence x_0, x_1, x_2, \dots . After multiplying by z and z^2 , we find that

$$\begin{aligned} zG(z) &= x_0z + x_1z^2 + x_2z^3 + \dots \\ z^2G(z) &= x_0z^2 + x_1z^3 + \dots \end{aligned}$$

Then we use the recurrence relation and the initial values to get

$$\begin{aligned} (1 - 2z + z^2)G(z) &= x_0 + (x_1 - 2x_0)z + 2z^2 + 3z^3 + 4z^4 + \dots \\ &= (1 - z) + z(1 + 2z + 3z^2 + 4z^3 + \dots) \\ &= (1 - z) + \frac{z}{(1 - z)^2}. \end{aligned}$$

Then

$$(1-z)^2 G(z) = (1-z) + \frac{z}{(1-z)^2}$$

so that a closed form is

$$G(z) = \frac{1}{1-z} + \frac{z}{(1-z)^4}.$$

Using the facts that

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad \text{and} \quad \frac{1}{(1-z)^4} = \sum_{k=0}^{\infty} \binom{k+3}{3} z^k$$

we obtain

$$\begin{aligned} G(z) &= \sum_{n=0}^{\infty} z^n + z \sum_{k=0}^{\infty} \binom{k+3}{3} z^k \\ &= \sum_{n=0}^{\infty} z^n + \sum_{k=0}^{\infty} \binom{k+3}{3} z^{k+1} \\ &= \sum_{n=0}^{\infty} z^n + \sum_{n=1}^{\infty} \binom{n+2}{3} z^n \end{aligned}$$

where we have used the substitution $n = k + 1$ in the second summation of the last line. The coefficient of z^n is x_n so that

$$x_n = 1 + \binom{n+2}{3}.$$