THE UNIVERSITY OF SYDNEY
MATH 1004

Second Semester	Discrete Mathematics	2011

Tutorial 12 Week 13

1. (*i*) Show that $x_n = 6 \cdot 2^n - 4$ is a solution to the recurrence relation

$$x_n = 3x_{n-1} - 2x_{n-2}.$$

(*ii*) Show that $x_n = (3^{n+1} - 1)/2$ is a solution to the recurrence relation

$$x_{n+1} - x_n = 3^{n+1}.$$

(*iii*) Show that $x_n = n!$ is a solution to the recurrence relation

$$x_n - n(n-1)x_{n-2} = 0.$$

Solution.

(*i*) If $x_n = 6 \cdot 2^n - 4$ then $x_{n-1} = 6 \cdot 2^{n-1} - 4$ and $x_{n-2} = 6 \cdot 2^{n-2} - 4$ so that

$$\begin{aligned} 3x_{n-1} - 2x_{n-2} &= 3(6 \cdot 2^{n-1} - 4) - 2(6 \cdot 2^{n-2} - 4) \\ &= 3 \cdot 6 \cdot 2^{n-1} - 12 - 2 \cdot 6 \cdot 2^{n-2} + 8 \\ &= 9 \cdot 2^n - 3 \cdot 2^n - 4 \\ &= 6 \cdot 2^n - 4 = x_n. \end{aligned}$$

Hence $x_n = 6 \cdot 2^n - 4$ is a solution of the recurrence relation.

(*ii*) If $x_n = (3^{n+1} - 1)/2$, then $x_{n+1} = (3^{n+2} - 1)/2$ so that

$$x_{n+1} - x_n = \frac{3^{n+2} - 1}{2} - \frac{3^{n+1} - 1}{2} = \frac{3^{n+2} - 3^{n+1}}{2} = 3^{n+1}.$$

Hence $x_n = (3^{n+1} - 1)/2$ is a solution of the recurrence relation.

(*iii*) If $x_n = n!$ then $x_{n-2} = (n-2)!$ and so

$$x_n - n(n-1)x_{n-2} = n! - n(n-1)(n-2)! = n! - n! = 0.$$

Hence $x_n = n!$ is a solution to the recurrence relation.

2. Solve the following recurrence relations:

(*i*) $x_n - 5x_{n-1} + 6x_{n-2} = 0,\ n \geq 2,\ x_0 = 3,\ x_1 = 7$

(*ii*) $x_n - 8x_{n-1} + 16x_{n-2} = 0,\ n \geq 2,\ x_0 = 3,\ x_1 = 20\,.$

Solution.

(*i*) The characteristic equation is $\lambda^2 - 5\lambda + 6 = 0$. That is, $(\lambda - 2)(\lambda - 3) = 0$ and so $\lambda = 2$ or $\lambda = 3$. Thus the general solution is

$$x_n = A2^n + B3^n,$$

for some constants A and B. Using the initial conditions $x_0 = 3$ and $x_1 = 7$, we obtain $3 = A + B$ and $7 = 2A + 3B$. Solving yields $A = 2$ and $B = 1$. Hence the solution is

$$x_n = 2 \cdot 2^n + 3^n.$$

(*ii*) The characteristic equation is $\lambda^2 - 8\lambda + 16 = 0$. That is, $(\lambda - 4)^2 = 0$ and so $\lambda = 4$ is a repeated root. Thus the general solution is

$$x_n = A4^n + Bn4^n,$$

for some constants A and B. Using the initial conditions $x_0 = 3$ and $x_1 = 20$, we obtain $3 = A$ and $20 = 4A + 4B$. Then $A = 3$ and $B = 2$. Hence the solution is

$$x_n = 3 \cdot 4^n + 2n\,4^n = (3 + 2n)4^n.$$

3. Solve the following recurrence relations:

(*i*) $x_n = 10x_{n-1} - 25x_{n-2}$, for $n \geq 2$, where $x_0 = -1$ and $x_1 = 5$.

(*ii*) $x_{n+3} - 6x_{n+2} + 11x_{n+1} - 6x_n = 0$, for $n \geq 0$, where $x_0 = 1$, $x_1 = 0$ and $x_2 = -1$.

Solution.

(*i*) The recurrence relation can be written as

$$x_n - 10x_{n-1} + 25x_{n-2} = 0.$$

Then the characteristic equation is

$$\lambda^2 - 10\lambda + 25 = 0.$$

That is $(\lambda - 5)^2 = 0$ so that $\lambda = 5$ is a repeated root with multiplicity 2. Thus the general solution to the recurrence relation is

$$x_n = A5^n + Bn5^n = (A + Bn)5^n,$$

where A and B are some constants. Now, using the initial values $x_0 = -1$ and $x_1 = 5$, we have $-1 = A$ and $5 = (A + B)5$. Solving yields $A = -1$ and $B = 2$. Hence the solution is

$$x_n = (-1 + 2n)5^n.$$

(*ii*) The characteristic equation is

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0.$$

Then $(\lambda - 1)(\lambda - 2)(\lambda - 3) = 0$, and so $\lambda = 1$, $\lambda = 2$ or $\lambda = 3$. Thus the general solution to the recurrence relation is

$$x_n = A + B2^n + C3^n,$$

for some constants where A, B and C. Now, using the initial values $x_0 = 1, x_1 = 0, x_2 = -1$, we obtain

$$\begin{aligned} A + \ \ B + \ \ C &= 1 \\ A + 2B + 3C &= 0 \\ A + 4B + 9C &= -1. \end{aligned}$$

Solving this system of linear equations yields $A = \frac{5}{2}$, $B = -2$, and $C = \frac{1}{2}$. Hence the solution is

$$x_n = \frac{5}{2} - 2 \cdot 2^n + \frac{1}{2} \cdot 3^n.$$

4. Solve the following recurrence relations:

(*i*) $x_n = 4x_{n-1} - 3x_{n-2}$, where $x_0 = 1$ and $x_1 = 2$.

(*ii*) $x_n = 3x_{n-1} - 3x_{n-2} + x_{n-3}$, where $x_0 = 0$, $x_1 = 1$ and $x_2 = 3$.

Solution.

(*i*) The recurrence relation can be written as

$$x_n - 4x_{n-1} + 3x_{n-2} = 0.$$

Then the characteristic equation is

$$\lambda^2 - 4\lambda + 3 = 0.$$

That is, $(\lambda - 1)(\lambda - 3) = 0$ so that $\lambda = 1$ or $\lambda = 3$. Thus the general solution for the given recurrence relation is

$$x_n = A1^n + B3^n,$$

for some constants A and B. Now, using the initial values $x_0 = 1$ and $x_1 = 2$, we obtain $1 = A + B$ and $2 = A + 3B$ which implies that $A = B = \frac{1}{2}$. Hence the solution is

$$x_n = \frac{1}{2} + \frac{1}{2} \cdot 3^n.$$

(ii) The recurrence relation can be written as

$$x_n - 3x_{n-1} + 3x_{n-2} - x_{n-3} = 0.$$

Then the characteristic equation is

$$\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0.$$

That is, $(\lambda - 1)^3 = 0$ so that $\lambda = 1$ is a repeated root with multiplicity 3. Thus the general solution for the recurrence relation is

$$x_n = A1^n + Bn1^n + Cn^2 1^n = A + Bn + Cn^2,$$

where A, B and C are some constants. Now, using the initial values $x_0 = 0$, $x_1 = 1$ and $x_2 = 3$, we have

$$\begin{aligned} 0 &= A \\ 1 &= A + B + C \\ 3 &= A + 2B + 4C. \end{aligned}$$

Solving the system yields $A = 0$, $B = C = \frac{1}{2}$. Hence the solution is

$$x_n = \frac{1}{2}(n + n^2) = \frac{1}{2}n(n+1).$$

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Problem Set 12

1. Solve the following recurrence relations:

(*i*) $x_n + 6x_{n-1} - 7x_{n-2} = 0$ for all $n \geq 2$, with $x_0 = 1$ and $x_1 = 2$.

(*ii*) $x_{n+3} - 4x_{n+2} + 5x_{n+1} - 2x_n = 0$ for all $n \geq 0$, with $x_0 = 4$, $x_1 = 7$ and $x_2 = 17$.

Solution.

(*i*) The characteristic equation is

$$\lambda^2 + 6\lambda - 7 = 0.$$

That is, $(\lambda - 1)(\lambda + 7) = 0$ and so $\lambda = 1$ or $\lambda = -7$. Thus the general solution is

$$x_n = A1^n + B(-7)^n,$$

for some constants A and B. Using the initial conditions $x_0 = 1$ and $x_1 = 2$, we obtain $1 = A + B$ and $2 = A - 7B$. Then $A = \frac{9}{8}$ and $B = -\frac{1}{8}$. Hence the solution is

$$x_n = \tfrac{9}{8} - \tfrac{1}{8}(-7)^n.$$

(*ii*) The characteristic equation is

$$\lambda^3 - 4\lambda^2 + 5\lambda - 2 = 0.$$

That is, $(\lambda - 2)(\lambda - 1)^2 = 0$ and so $\lambda = 2$ or $\lambda = 1$, a double root. Thus the general solution is

$$x_n = A2^n + (B + Cn)1^n,$$

for some constants A, B and C. Using the initial conditions $x_0 = 4$, $x_1 = 7$ and $x_2 = 17$, we obtain

$$\begin{aligned} 4 &= A + B, \\ 7 &= 2A + B + C, \\ 17 &= 4A + B + 2C. \end{aligned}$$

Solving this system of linear equations yields $A = 7$, $B = -3$ and $C = -4$. Hence the solution is

$$x_n = 7 \cdot 2^n + (-3 - 4n)1^n = 7 \cdot 2^n - 3 - 4n.$$

2. Solve the following recurrence relations:

(*i*) $x_{n+2} - 4x_{n+1} + 4x_n = 0$ for all $n \geq 0$, with $x_0 = 1$ and $x_1 = 4$.

(*ii*) $x_{n+2} + 2x_{n+1} - 3x_n = 0$ for all $n \geq 0$, with $x_0 = 1$ and $x_1 = -1$.

Solution.

(*i*) The characteristic equation is $\lambda^2 - 4\lambda + 4 = 0$. That is, $(\lambda - 2)^2 = 0$ and so $\lambda = 2$ is a repeated root (root of multiplicity 2). Thus the general solution is

$$x_n = (A + Bn)2^n,$$

for some constants A and B. Using the initial conditions $x_0 = 1$ and $x_1 = 4$, we obtain $1 = A$ and $4 = 2(A + B)$. Then $A = 1$ and $B = 1$. Hence the solution is

$$x_n = (1 + n)2^n.$$

(*ii*) The characteristic equation is $\lambda^2 + 2\lambda - 3 = 0$. That is, $(\lambda - 1)(\lambda + 3) = 0$ and so $\lambda = 1$ or $\lambda = -3$. Thus the general solution is

$$x_n = A1^n + B(-3)^n,$$

for some constants A and B. Using the initial conditions $x_0 = 1$ and $x_1 = -1$, we obtain $1 = A + B$ and $-1 = A - 3B$. Solving yields $A = \frac{1}{2}$ and $B = \frac{1}{2}$. Hence the solution is

$$x_n = \tfrac{1}{2} + \tfrac{1}{2}(-3)^n.$$

3. Solve the following recurrence relations:

(*i*) $x_n = -x_{n-1} + 6x_{n-2}$, where $x_0 = 7$ and $x_1 = 4$.

(*ii*) $x_n = 6x_{n-1} - 9x_{n-2}$, where $x_0 = -2$ and $x_1 = 6$.

Solution.

(*i*) The recurrence relation can be written as

$$x_n + x_{n-1} - 6x_{n-2} = 0.$$

Then the characteristic equation is

$$\lambda^2 + \lambda - 6 = 0.$$

That is, $(\lambda + 3)(\lambda - 2) = 0$ so that $\lambda = -3$ or $\lambda = 2$. Thus the general solution to the recurrence relation is

$$x_n = A(-3)^n + B2^n,$$

for some constants A and B. Now, using the initial values $x_0 = 7$ and $x_1 = 4$, we have $7 = A + B$ and $4 = -3A + 2B$. Solving yields $A = 2$ and $B = 5$. Hence the solution is

$$x_n = 2(-3)^n + 5 \cdot 2^n.$$

(*ii*) The recurrence relation can be written as

$$x_n - 6x_{n-1} + 9x_{n-2} = 0.$$

The characteristic equation is

$$\lambda^2 - 6\lambda + 9 = 0.$$

That is, $(\lambda - 3)^2 = 0$ so that $\lambda = 3$ is a repeated root with multiplicity 2. Thus the general solution to the recurrence relation is

$$x_n = (A + Bn)3^n,$$

where A and B are some constants. Now, using the initial values $x_0 = -2$ and $x_1 = 6$, we obtain $-2 = A$ and $6 = 3(A + B)$. Solving yields $A = -2$ and $B = 4$ and the solution is

$$x_n = -2 \cdot 3^n + 4n \cdot 3^n = (-2 + 4n)3^n.$$

4. Solve the following recurrence relations:

(*i*) $x_n = 5x_{n-1} - 6x_{n-2}$, where $x_0 = 1$ and $x_1 = 1$.

(*ii*) $x_n - 5x_{n-1} + 8x_{n-2} - 4x_{n-3} = 0$, where $x_0 = 0$, $x_1 = 2$ and $x_2 = 4$.

Solution.

(*i*) The recurrence relation can be written as

$$x_n - 5x_{n-1} + 6x_{n-2} = 0.$$

Then the characteristic equation is

$$\lambda^2 - 5\lambda + 6 = 0.$$

That is, $(\lambda - 2)(\lambda - 3) = 0$ so that $\lambda = 2$ or $\lambda = 3$. Thus the general solution to the recurrence relation is

$$x_n = A2^n + B3^n,$$

for some constants A and B. Now, using the initial values $x_0 = 1$ and $x_1 = 1$, we have $1 = A + B$ and $1 = 2A + 3B$. A simple calculation gives $A = 2$ and $B = -1$. Hence the solution is

$$x_n = 2 \cdot 2^n - 3^n.$$

(*ii*) The characteristic equation of the recurrence relation is

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0.$$

Then $(\lambda-1)(\lambda-2)^2 = 0$ so that $\lambda = 1$ is a root and $\lambda = 2$ is a repeated root with multiplicity 2. Thus the general solution for the recurrence relation is

$$x_n = A1^n + B2^n + Cn2^n,$$

for some constants A, B and C. Now, using the initial values $x_0 = 0$, $x_1 = 2$ and $x_2 = 4$, we obtain

$$\begin{aligned} 0 &= A + B \\ 2 &= A + 2B + 2C \\ 4 &= A + 4B + 8C. \end{aligned}$$

Solving this system of linear equations yields $A = -4$, $B = 4$ and $C = -1$. Hence the solution is

$$x_n = -4 + 4 \cdot 2^n - n \cdot 2^n.$$