The University of Sydney
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Assignment Solutions

MATH1014: Introduction to Linear Algebra — Semester 2, 2009

1. (a) The point $(7, -1, -1)$ does not lie on l, since the equations

$$\begin{aligned} 3 + 2t &= 7 \\ -1 + t &= -1 \\ 5 - 3t &= -1 \end{aligned}$$

are inconsistent.

(b) The lines are not parallel. The line l is in the direction of the vector $\begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$, and the line m is in the direction of the vector $\begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix}$. Since these two vectors are not parallel, the lines are not parallel.

(c) A line perpendicular to the plane P would be in the direction of the normal to P, which is $\begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$. Since l is not in this direction it is not perpendicular to P.

(d) The line m passes through the point $(-1, -1, 1)$ and is parallel to $\begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix}$.
The point $(-1, -1, 1)$ lies in the plane P, since $2 \times -1 - 3 \times -1 + 5 \times 1 = 6$.
The dot product $\begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix} = 0$. Hence, the line is perpendicular to the normal to P, and therefore parallel to P.
So m is parallel to P, and passes through a point that lies in P. Hence m lies in P.

(e) In order to find the point of intersection, we substitute the arbitrary point $(3 + 2t, -1 + t, 5 - 3t)$ on l into the equation of P, and solve for t.

$$\begin{aligned} 2(3 + 2t) - 3(-1 + t) + 5(5 - 3t) &= 6 \\ 14t &= 28 \\ t &= 2. \end{aligned}$$

The point of intesection is, therefore, $(7, 1, -1)$.

2. (a)

$$\begin{aligned}
|A-\lambda I| &= \begin{vmatrix} 3-\lambda & 0 & -2 \\ 0 & 3-\lambda & 4 \\ 1 & 2 & 1-\lambda \end{vmatrix} \\
&= (3-\lambda)\begin{vmatrix} 3-\lambda & 4 \\ 2 & 1-\lambda \end{vmatrix} + \begin{vmatrix} 0 & -2 \\ 3-\lambda & 4 \end{vmatrix} \\
&= (3-\lambda)(\lambda^2-4\lambda-5)+2(3-\lambda) \\
&= (3-\lambda)(\lambda^2-4\lambda-3).
\end{aligned}$$

So $|A-\lambda I|=0$ when $\lambda = 3,\ 2\pm\sqrt{7}$.
That is, the eigenvalues of A are $3,\ 2+\sqrt{7},\ 2-\sqrt{7}$.

(b) $A-3I = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 0 & 4 \\ 1 & 2 & -2 \end{bmatrix}$, and this reduces to $\begin{bmatrix} 1 & 2 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

So the eigenvectors are $\begin{bmatrix} -2t \\ t \\ 0 \end{bmatrix}$, for all $t \in \mathbb{R}$, $t \neq 0$.

3. (a) Taking the states to be black, white, or grey-coated, in that order, the transition matrix is

$$P = \begin{bmatrix} 0.5 & 0.2 & 0.1 \\ 0.5 & 0.6 & 0.45 \\ 0 & 0.2 & 0.45 \end{bmatrix}.$$

(b) The initial state vector is $\mathbf{x}_0 = \begin{bmatrix} 0.5 \\ 0.5 \\ 0 \end{bmatrix}$. In the next generation, the state vector is

$$\mathbf{x}_1 = P\mathbf{x}_0 = \begin{bmatrix} 0.5 & 0.2 & 0.1 \\ 0.5 & 0.6 & 0.45 \\ 0 & 0.2 & 0.45 \end{bmatrix}\begin{bmatrix} 0.5 \\ 0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.35 \\ 0.55 \\ 0.10 \end{bmatrix}.$$

So 10% of the next generation will be grey-coated.

(c) We need to find the steady state vector. That is, we need to solve the equation $P\mathbf{x} = \mathbf{x}$, or $(P-I)\mathbf{x} = \mathbf{0}$. The augmented matrix for this system of equations is

$$\left[\begin{array}{ccc|c} -0.5 & 0.2 & 0.1 & 0 \\ 0.5 & -0.4 & 0.45 & 0 \\ 0 & 0.2 & -0.55 & 0 \end{array}\right],$$

and this reduces to

$$\left[\begin{array}{ccc|c} 1 & -0.4 & -0.2 & 0 \\ 0 & 1 & -2.75 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Hence a steady state vector is $\begin{bmatrix} 1.3t \\ 2.75t \\ t \end{bmatrix}$ ($t \in \mathbb{R}$, $t \neq 0$), and the percentages of the dog population with black, white and grey coats in the long run are 26%, 54% and 20% (rounded to the nearest whole number).

4. Firstly, note that in the calculation of the dot product $\mathbf{c} \cdot \mathbf{v}$, the digit in the n^{th} position of the SID $\mathbf{v}$, *counting from the right*, is multiplied by n, where $n = 1, 2, \ldots, 9$.

We consider two separate cases.

Case 1: The SID contains a 9 that has been incorrectly entered as a zero.

Let $\mathbf{v}$ be the correct SID, and $\mathbf{w}$ be the incorrectly entered SID. Suppose the 9 that has been entered incorrectly as a 0 is in the n^{th} position, counting from the right.

Now, the contribution to $\mathbf{c} \cdot \mathbf{v}$ from the 9 is $9n$, and the contribution to $\mathbf{c} \cdot \mathbf{w}$ from the 0 is zero, and $\mathbf{c} \cdot \mathbf{v} = 0$. So $\mathbf{c} \cdot \mathbf{w} = \mathbf{c} \cdot \mathbf{v} - 9n = -9n = 2n$ in $\mathbb{Z}_{11}$. Since $2n \neq 0$ in $\mathbb{Z}_{11}$ for any $n = 1, 2, \ldots, 9$, $\mathbf{c} \cdot \mathbf{w} \neq 0$ in $\mathbb{Z}_{11}$. Hence the error will be detected.

Case 2: The SID contains one digit that has been incorrectly entered as $x + 1$ instead of x, for $x = 0, 1, \ldots, 8$.

Let $\mathbf{v}$ be the correct SID, and $\mathbf{w}$ be the incorrectly entered SID. Suppose the x that has been entered incorrectly as $x + 1$ is in the n^{th} position, counting from the right.

In this case, the contribution to $\mathbf{c} \cdot \mathbf{v}$ from the x is $n \times x$, and the contribution to $\mathbf{c} \cdot \mathbf{w}$ from the $x + 1$ is $n(x + 1)$, and again $\mathbf{c} \cdot \mathbf{v} = 0$.
So $\mathbf{c} \cdot \mathbf{w} = \mathbf{c} \cdot \mathbf{v} - nx + n(x + 1) = n$, and $n \neq 0$ in $\mathbb{Z}_{11}$ for any $n = 1, 2, \ldots, 9$. Hence $\mathbf{c} \cdot \mathbf{w} \neq 0$ in $\mathbb{Z}_{11}$, and the error will be detected.