The University of Sydney
School of Mathematics and Statistics

Tutorial 2 (Week 2)

MATH1014: Introduction to Linear Algebra — Semester 2, 2009

Preparatory questions (attempt before the tutorial)

1. Let P be the point $(3, 1)$ and Q the point $(4, -2)$ in the xy-plane. As usual the origin $(0, 0)$ is denoted by O.

(a) Write down the position vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ as column vectors, and also in terms of $\mathbf{i}$ and $\mathbf{j}$.

(b) Write down the displacement vector $\overrightarrow{PQ}$ as a column vector, and also in terms of $\mathbf{i}$ and $\mathbf{j}$.

(c) Write down the coordinates of the point R such that $\overrightarrow{OR} = \overrightarrow{PQ}$.

(d) Find the length of $\overrightarrow{PQ}$.

2. Given points $A = (4, -1, 5)$ and $B = (6, -1, -2)$ in space, find

(a) the position vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ as column vectors, and also in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$;

(b) the displacement vector $\overrightarrow{AB}$ as a column vector, and also in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$;

(c) the unit vector pointing from A towards B;

(d) the unit vector pointing from B towards A.

Tutorial exercises

3. For each of the following vectors, find its length, and a unit vector pointing in the same direction.

(a) $\mathbf{u} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$ (b) $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix}$ (c) $\mathbf{a} = -\mathbf{i} + 10\,\mathbf{j}$ (d) $\mathbf{b} = 4\,\mathbf{i} - \mathbf{j} + 3\,\mathbf{k}$

4. Given that

$$\mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \qquad \mathbf{w} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix},$$

find

(a) $\mathbf{u} \cdot \mathbf{v}$ (b) $\mathbf{u} \cdot \mathbf{w}$ (c) $\mathbf{v} \cdot \mathbf{w}$ (d) $\mathbf{u} \cdot \mathbf{u}$ (e) $\mathbf{v} \cdot \mathbf{v}$ (f) $\mathbf{w} \cdot \mathbf{w}$
(g) $||\mathbf{u}||$ (h) $||\mathbf{v}||$ (i) $||\mathbf{w}||$ (j) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w})$ (k) $\mathbf{u} \cdot (\mathbf{v} - \mathbf{w})$

5. Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be as in question **4**. Let α be the angle between $\mathbf{u}$ and $\mathbf{v}$, β be the angle between $\mathbf{u}$ and $\mathbf{w}$, and γ the angle between $\mathbf{v}$ and $\mathbf{w}$. Find

(a) $\cos\alpha$ (b) $\cos\beta$ (c) $\cos\gamma$

6. Find any values of k for which the vectors $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} k-1 \\ k \\ k+1 \end{bmatrix}$ are perpendicular.

7. Show that the vectors $\mathbf{u} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} k \\ 2k \end{bmatrix}$ are perpendicular for all values of k. Draw a diagram to illustrate this result.

8. Let $A = (-3, 2)$, $B = (1, 0)$ and $C = (4, 6)$ be points in the plane.
Prove that ΔABC is a right-angled triangle.

9. Given that $\mathbf{a} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{c} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ find

(a) $\mathbf{a} \times \mathbf{b}$ (b) $\mathbf{a} \times \mathbf{c}$ (c) $\mathbf{b} \times \mathbf{c}$ (d) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ (e) $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$
(f) $\mathbf{a} \times (\mathbf{a} \times \mathbf{c})$ (g) $\mathbf{a} \times (\mathbf{a} + \mathbf{c})$ (h) $(\mathbf{a} \times \mathbf{a}) \times \mathbf{c}$ (i) $\mathbf{a} \times (\mathbf{b} - 2\mathbf{c})$

10. Find two unit vectors perpendicular to both $\mathbf{v}$ and $\mathbf{w}$ where

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ -7 \end{bmatrix} \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}.$$

Further exercises

In addition to these exercises, the following exercises from the textbook – *Linear Algebra: A Modern Introduction* by David Poole – should be attempted:
Exercises 1.2: 1, 3, 5, 7, 9, 11, 17, 19, 25, 31, 43, 45, 61.

Answers to selected exercises

1. (a) $\overrightarrow{OP} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} = 3\mathbf{i} + \mathbf{j}$, $\overrightarrow{OQ} = \begin{bmatrix} 4 \\ -2 \end{bmatrix} = 4\mathbf{i} - 2\mathbf{j}$ (b) $\overrightarrow{PQ} = \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \mathbf{i} - 3\mathbf{j}$

(c) $R = (1, -3)$ (d) $||\overrightarrow{PQ}|| = \sqrt{10}$

2. (a) $\overrightarrow{OA} = \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix} = 4\mathbf{i} - \mathbf{j} + 5\mathbf{k}$, $\overrightarrow{OB} = \begin{bmatrix} 6 \\ -1 \\ -2 \end{bmatrix} = 6\mathbf{i} - \mathbf{j} - 2\mathbf{k}$.

(b) $\overrightarrow{AB} = \begin{bmatrix} 2 \\ 0 \\ -7 \end{bmatrix} = 2\mathbf{i} - 7\mathbf{k}$.

(c) $\frac{1}{\sqrt{53}}\begin{bmatrix}2\\0\\-7\end{bmatrix}$ $\left(\text{or } \frac{1}{\sqrt{53}}(2\mathbf{i}-7\mathbf{k})\right)$.

(d) $-\frac{1}{\sqrt{53}}\begin{bmatrix}2\\0\\-7\end{bmatrix}$ $\left(\text{or } -\frac{1}{\sqrt{53}}(2\mathbf{i}-7\mathbf{k})\right)$.

3. (a) $\|\mathbf{u}\| = 2\sqrt{2}$; unit vector in the direction of $\mathbf{u} = \begin{bmatrix}\frac{1}{\sqrt{2}}\\ \\ \frac{-1}{\sqrt{2}}\end{bmatrix}$

(b) $\|\mathbf{v}\| = \sqrt{21}$; unit vector in the direction of $\mathbf{v} = \begin{bmatrix}\frac{1}{\sqrt{21}}\\ \frac{2}{\sqrt{21}}\\ \frac{-4}{\sqrt{21}}\end{bmatrix}$

(c) $\|\mathbf{a}\| = \sqrt{101}$; unit vector in the direction of $\mathbf{a} = -\frac{1}{\sqrt{101}}\mathbf{i} + \frac{10}{\sqrt{101}}\mathbf{j}$.

(d) $\|\mathbf{b}\| = \sqrt{26}$; unit vector in the direction of $\mathbf{b} = \frac{4}{\sqrt{26}}\mathbf{i} - \frac{1}{\sqrt{26}}\mathbf{j} + \frac{3}{\sqrt{26}}\mathbf{k}$.

4. (a) 6 (b) 5 (c) 1 (d) 6 (e) 9 (f) 10 (g) $\sqrt{6}$ (h) 3
(i) $\sqrt{10}$ (j) 11 (k) 1

5. (a) $\frac{\sqrt{6}}{3}$ (b) $\frac{\sqrt{15}}{6}$ (c) $\frac{\sqrt{10}}{30}$

6. $k = -\frac{1}{3}$.

9. (a) $\begin{bmatrix}-1\\-2\\3\end{bmatrix}$ (b) $\begin{bmatrix}-1\\-2\\-2\end{bmatrix}$ (c) $\begin{bmatrix}1\\-3\\2\end{bmatrix}$ (d) $\begin{bmatrix}-2\\-4\\-5\end{bmatrix}$ (e) $\begin{bmatrix}-2\\-5\\-4\end{bmatrix}$ (f) $\begin{bmatrix}2\\4\\-5\end{bmatrix}$

(g) $\begin{bmatrix}-1\\-2\\-2\end{bmatrix}$ (h) $\begin{bmatrix}0\\0\\0\end{bmatrix}$ (i) $\begin{bmatrix}1\\2\\7\end{bmatrix}$

10. $\pm\frac{\sqrt{2}}{6}\begin{bmatrix}1\\-4\\-1\end{bmatrix}$