The University of Sydney
School of Mathematics and Statistics

Tutorial 2 (Week 2)

MATH1014: Introduction to Linear Algebra — Semester 2, 2009

Preparatory questions (attempt before the tutorial)

1. Let P be the point $(3, 1)$ and Q the point $(4, -2)$ in the xy-plane. As usual the origin $(0, 0)$ is denoted by O.

(a) Write down the position vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ as column vectors, and also in terms of $\mathbf{i}$ and $\mathbf{j}$.

(b) Write down the displacement vector $\overrightarrow{PQ}$ as a column vector, and also in terms of $\mathbf{i}$ and $\mathbf{j}$.

(c) Write down the coordinates of the point R such that $\overrightarrow{OR} = \overrightarrow{PQ}$.

(d) Find the length of $\overrightarrow{PQ}$.

2. Given points $A = (4, -1, 5)$ and $B = (6, -1, -2)$ in space, find

(a) the position vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ as column vectors, and also in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$;

(b) the displacement vector $\overrightarrow{AB}$ as a column vector, and also in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$;

(c) the unit vector pointing from A towards B;

(d) the unit vector pointing from B towards A.

Tutorial exercises

3. For each of the following vectors, find its length, and a unit vector pointing in the same direction.

(a) $\mathbf{u} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$ (b) $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix}$ (c) $\mathbf{a} = -\mathbf{i} + 10\mathbf{j}$ (d) $\mathbf{b} = 4\mathbf{i} - \mathbf{j} + 3\mathbf{k}$

4. Given that

$$\mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \qquad \mathbf{w} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix},$$

find

(a) $\mathbf{u} \cdot \mathbf{v}$ (b) $\mathbf{u} \cdot \mathbf{w}$ (c) $\mathbf{v} \cdot \mathbf{w}$ (d) $\mathbf{u} \cdot \mathbf{u}$ (e) $\mathbf{v} \cdot \mathbf{v}$ (f) $\mathbf{w} \cdot \mathbf{w}$
(g) $||\mathbf{u}||$ (h) $||\mathbf{v}||$ (i) $||\mathbf{w}||$ (j) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w})$ (k) $\mathbf{u} \cdot (\mathbf{v} - \mathbf{w})$

5. Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be as in question **4**. Let α be the angle between $\mathbf{u}$ and $\mathbf{v}$, β be the angle between $\mathbf{u}$ and $\mathbf{w}$, and γ the angle between $\mathbf{v}$ and $\mathbf{w}$. Find

(a) $\cos\alpha$ (b) $\cos\beta$ (c) $\cos\gamma$

6. Find any values of k for which the vectors $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} k-1 \\ k \\ k+1 \end{bmatrix}$ are perpendicular.

7. Show that the vectors $\mathbf{u} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} k \\ 2k \end{bmatrix}$ are perpendicular for all values of k. Draw a diagram to illustrate this result.

8. Let $A = (-3, 2)$, $B = (1, 0)$ and $C = (4, 6)$ be points in the plane.
Prove that ΔABC is a right-angled triangle.

9. Given that $\mathbf{a} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{c} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ find

(a) $\mathbf{a} \times \mathbf{b}$ (b) $\mathbf{a} \times \mathbf{c}$ (c) $\mathbf{b} \times \mathbf{c}$ (d) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ (e) $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$
(f) $\mathbf{a} \times (\mathbf{a} \times \mathbf{c})$ (g) $\mathbf{a} \times (\mathbf{a} + \mathbf{c})$ (h) $(\mathbf{a} \times \mathbf{a}) \times \mathbf{c}$ (i) $\mathbf{a} \times (\mathbf{b} - 2\mathbf{c})$

10. Find two unit vectors perpendicular to both $\mathbf{v}$ and $\mathbf{w}$ where

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ -7 \end{bmatrix} \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}.$$

Further exercises

In addition to these exercises, the following exercises from the textbook – *Linear Algebra: A Modern Introduction* by David Poole – should be attempted:
Exercises 1.2: 1, 3, 5, 7, 9, 11, 17, 19, 25, 31, 43, 45, 61.

Solutions

1. (a) $\overrightarrow{OP} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} = 3\mathbf{i} + \mathbf{j}$, $\overrightarrow{OQ} = \begin{bmatrix} 4 \\ -2 \end{bmatrix} = 4\mathbf{i} - 2\mathbf{j}$

(b) $\overrightarrow{PQ} = \begin{bmatrix} 4-3 \\ -2-1 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \mathbf{i} - 3\mathbf{j}$

(c) If $\overrightarrow{OR}$ is the positon vector $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$, then the point R is $(1, -3)$.

(d) $||\overrightarrow{PQ}|| = \sqrt{1^2 + 3^2} = \sqrt{10}$

2. (a) $\overrightarrow{OA} = 4\mathbf{i} - \mathbf{j} + 5\mathbf{k}$, $\overrightarrow{OB} = 6\mathbf{i} - \mathbf{j} - 2\mathbf{k}$.

(b) $\overrightarrow{AB} = \begin{bmatrix} 6-4 \\ -1-(-1) \\ -2-5 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -7 \end{bmatrix} = 2\mathbf{i} - 7\mathbf{k}$.

(c) $||\overrightarrow{AB}|| = \sqrt{2^2 + (-7)^2} = \sqrt{53}$, and so the unit vector pointing from A towards B is $\frac{1}{\sqrt{53}} \begin{bmatrix} 2 \\ 0 \\ -7 \end{bmatrix}$ $\left(\text{or } \frac{1}{\sqrt{53}}(2\mathbf{i} - 7\mathbf{k})\right)$.

(d) The unit vector pointing from B towards A is simply the negative of the vector found in part (c) – that is, $-\frac{1}{\sqrt{53}} \begin{bmatrix} 2 \\ 0 \\ -7 \end{bmatrix}$ $\left(\text{or } -\frac{1}{\sqrt{53}}(2\mathbf{i} - 7\mathbf{k})\right)$.

3. (a) $||\mathbf{u}|| = \sqrt{2^2 + (-2)^2} = 2\sqrt{2}$.

Hence, a unit vector in the direction of $\mathbf{u}$ is $\frac{1}{2\sqrt{2}} \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$.

(b) $||\mathbf{v}|| = \sqrt{1^2 + 2^2 + (-4)^2} = \sqrt{21}$.

Hence, a unit vector in the direction of $\mathbf{v}$ is $\begin{bmatrix} \frac{1}{\sqrt{21}} \\ \frac{2}{\sqrt{21}} \\ \frac{-4}{\sqrt{21}} \end{bmatrix}$.

(c) $\mathbf{a} = -\mathbf{i} + 10\mathbf{j} = \begin{bmatrix} -1 \\ 10 \end{bmatrix}$, so $||\mathbf{a}|| = \sqrt{(-10^2 + 10^2} = \sqrt{101}$, and a unit vector in the direction of $\mathbf{a}$ is $-\frac{1}{\sqrt{101}}\mathbf{i} + \frac{10}{\sqrt{101}}\mathbf{j}$.

(d) $\mathbf{b} = 4\mathbf{i} - \mathbf{j} + 3\mathbf{k} = \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}$, so $||\mathbf{b}|| = \sqrt{4^2 + (-1)^2 + 3^2} = \sqrt{26}$, and a unit vector in the direction of $\mathbf{b}$ is $\frac{4}{\sqrt{26}}\mathbf{i} - \frac{1}{\sqrt{26}}\mathbf{j} + \frac{3}{\sqrt{26}}\mathbf{k}$.

4. (a) $\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = 2 \times 1 + (-1) \times (-2) + 1 \times 2 = 6.$

(b) $\mathbf{u} \cdot \mathbf{w} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} = 2 \times 3 + (-1) \times 0 + 1 \times (-1) = 5.$

(c) $\mathbf{v} \cdot \mathbf{w} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} = 1 \times 3 + (-2) \times 0 + 2 \times (-1) = 1.$

(d) $\mathbf{u} \cdot \mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = 2 \times 2 + (-1) \times (-1) + 1 \times 1 = 6.$

(e) $\mathbf{v} \cdot \mathbf{v} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = 1 \times 1 + (-2) \times (-2) + 2 \times 2 = 9.$

(f) $\mathbf{w} \cdot \mathbf{w} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} = 3 \times 3 + 0 \times 0 + (-1) \times (-1) = 10.$

(g) $||\mathbf{u}|| = \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{6}.$

(h) $||\mathbf{v}|| = \sqrt{1^2 + (-2)^2 + 2^2} = 3.$

(i) $||\mathbf{w}|| = \sqrt{3^2 + (-1)^2} = \sqrt{10}.$

(j) $\mathbf{u} \cdot \big(\mathbf{v} + \mathbf{w}\big) = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} = 2 \times 4 + (-1) \times (-2) + 1 \times 1 = 11.$

(k) $\mathbf{u} \cdot \big(\mathbf{v} - \mathbf{w}\big) = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ -2 \\ 3 \end{bmatrix} = 2 \times (-2) + (-1) \times (-2) + 1 \times 3 = 1.$

5. (a) $\cos\alpha = \dfrac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}||||\mathbf{v}||} = \dfrac{6}{\sqrt{6} \times 3} = \dfrac{\sqrt{6}}{3}.$

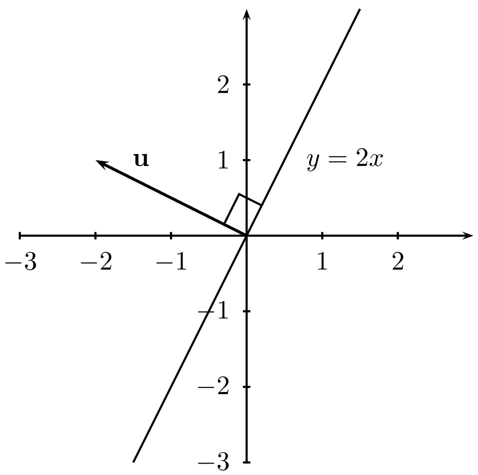
(b) $\cos\beta = \dfrac{\mathbf{u} \cdot \mathbf{w}}{||\mathbf{u}||||\mathbf{w}||} = \dfrac{5}{\sqrt{6} \times \sqrt{10}} = \dfrac{\sqrt{15}}{6}.$

(c) $\cos\gamma = \dfrac{\mathbf{v} \cdot \mathbf{w}}{||\mathbf{v}||||\mathbf{w}||} = \dfrac{1}{3 \times \sqrt{10}} = \dfrac{\sqrt{10}}{30}.$

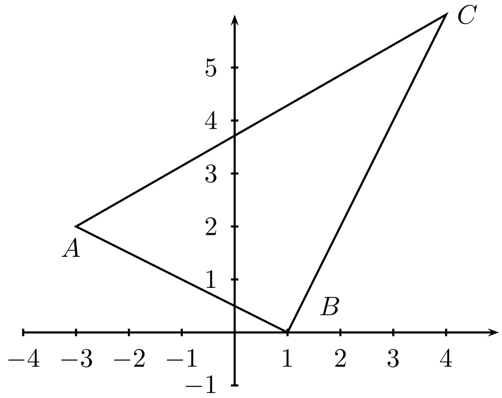
6. The vectors are perpendicular if their dot product is zero. So $\mathbf{u}$ and $\mathbf{v}$ are perpendicular if $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} k-1 \\ k \\ k+1 \end{bmatrix} = k - 1 + 2k + 3k + 3 = 6k + 2 = 0$. That is, if $k = -\dfrac{1}{3}$.

7. $\mathbf{u} \cdot \mathbf{v} = -2k + 2k = 0$ for all k. Therefore, $\mathbf{u}$ and $\mathbf{v}$ are perpendicular for all k.

The diagram shows the vector $\mathbf{u}$ and the straight line $y = 2x$, to which $\mathbf{u}$ is perpendicular. For any value of k the vector $\mathbf{v}$, drawn as a position vector, will lie along the line $y = 2x$.



8.



From the diagram, it certainly appears that the angle at B is a right angle. We can prove that this is the case by showing that $\overrightarrow{BA} \cdot \overrightarrow{BC} = 0$.

Now, $\overrightarrow{BA} = -4\ \mathbf{i} + 2\ \mathbf{j}$ and $\overrightarrow{BC} = 3\ \mathbf{i} + 6\ \mathbf{j}$, so $\overrightarrow{BA} \cdot \overrightarrow{BC} = -12 + 12 = 0$ as required.

9. (a) Using the method explained in lectures (and on page 45 of the textbook), we have:

2	1	
−1	1	
0	1	$-1 \times 1 - 0 \times 1 = -1$
2	1	$0 \times 1 - 2 \times 1 = -2$
−1	1	$2 \times 1 - (-1 \times 1) = 3$

Hence, $\mathbf{a} \times \mathbf{b} = \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix}$.

(b) Using the same method again:

2	−2	
−1	0	
0	1	$-1 \times 1 - 0 \times 0 = -1$
2	−2	$0 \times -2 - 2 \times 1 = -2$
−1	0	$2 \times 0 - (-1 \times -2) = -2$

Hence, $\mathbf{a} \times \mathbf{c} = \begin{bmatrix} -1 \\ -2 \\ -2 \end{bmatrix}$.

Use the same method for calculating the cross products in parts (c) – (i).

(c) $\mathbf{b} \times \mathbf{c} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$

(d) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \\ -5 \end{bmatrix}$

(e) $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix} \times \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -5 \\ -4 \end{bmatrix}$

(f) $\mathbf{a} \times (\mathbf{a} \times \mathbf{c}) = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \times \begin{bmatrix} -1 \\ -2 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ -5 \end{bmatrix}$

(g) $\mathbf{a} \times (\mathbf{a} + \mathbf{c}) = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ -2 \end{bmatrix}$

(h) $(\mathbf{a} \times \mathbf{a}) \times \mathbf{c} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(i) $\mathbf{a} \times (\mathbf{b} - 2\mathbf{c}) = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 5 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$

10. The cross product $\mathbf{v} \times \mathbf{w}$ is perpendicular to both $\mathbf{v}$ and $\mathbf{w}$. Calculating the cross product:

$$\begin{array}{ccl} 1 & 5 & \\ 2 & 1 & \\ -7 & 1 & 2 \times 1 - (-7 \times 1) = 9 \\ 1 & 5 & -7 \times 5 - 1 \times 1 = -36 \\ 2 & 1 & 1 \times 1 - 2 \times 5 = -9 \end{array}$$

Hence, $\mathbf{v} \times \mathbf{w} = \begin{bmatrix} 9 \\ -36 \\ -9 \end{bmatrix}$, and

$||\mathbf{v} \times \mathbf{w}|| = \sqrt{9^2 + (-36)^2 + (-9)^2} = 9\sqrt{1 + 16 + 1} = 9\sqrt{18} = 27\sqrt{2}.$

So two unit vectors perpendicular to both $\mathbf{v}$ and $\mathbf{w}$ are $\pm\frac{1}{27\sqrt{2}} \begin{bmatrix} 9 \\ -36 \\ -9 \end{bmatrix}$,

or $\begin{bmatrix} \sqrt{2}/6 \\ -2\sqrt{2}/3 \\ -\sqrt{2}/6 \end{bmatrix}$ and $\begin{bmatrix} -\sqrt{2}/6 \\ 2\sqrt{2}/3 \\ \sqrt{2}/6 \end{bmatrix}$.