THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Tutorial 3 (Week 3)

MATH1014: Introduction to Linear Algebra — Semester 2, 2009

Preparatory questions (attempt before the tutorial)

1. (a) Find a vector parallel to each of the following straight lines. (Hint: choose any two points on the line and find the vector joining them.)

(i) $y = 2x$ (ii) $x = 2y$ (iii) $2x + y = 0$ (iv) $x + 2y = 0$
(v) $x + y = 2$ (vi) $y = 2x - 3$ (vii) $y = 3 - 2x$ (viii) $2y = x + 5$
(ix) $2y = 5 - x$ (x) $x - 2y = 3$ (xi) $2x + y + 1 = 0$

(b) Which of the lines in part (a) are parallel to the vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$?

Draw a sketch to illustrate your answer.

(c) Which of the lines in part (a) are perpendicular to $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$?

2. (a) Find a vector normal to the plane $x + y + 5z = 6$.

(b) Find a vector normal to the plane $z = 7 - 3x + 8y$.

(c) Find a unit vector normal to the plane $2x + y - z = 6$.

Tutorial exercises

3. Find a vector equation, parametric equations, and a Cartesian equation for the straight line through the point $(0, 0)$ parallel to the vector $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$.

4. Find a vector equation, parametric equations, and a Cartesian equation for the straight line through the point $(1, 4)$ parallel to the vector $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$.

5. Find a vector equation, and parametric equations, for the line passing through the point $(1, 0, -1)$ in the direction of the vector $\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$.

6. Find a vector equation, and parametric equations, for the line passing through $P = (-4, 3, 5)$ and $Q = (-2, 4, -1)$.

7. A plane $\mathcal{P}$ contains the point $(2, 3, 5)$ and has normal vector $\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$. Find the equation of $\mathcal{P}$ in normal form, and hence find a Cartesian equation for $\mathcal{P}$.

8. Find the Cartesian equation of the plane containing the points $P = (1, 2, 3)$, $Q = (-1, -2, -3)$ and $R = (4, -4, 4)$.

9. Find parametric equations for the line passing through $(1, 0, -2)$ and perpendicular to the plane $3x - 4y + z = 6$.

10. Show that the line with parametric equations

$$x = 3 + 2t, \quad y = 4 + 3t, \quad z = 5 + 4t \quad (t \in \mathbb{R})$$

is parallel to the plane $4x + 4y - 5z = 14$.

Further exercises

In addition to these exercises, the following exercises from the textbook – *Linear Algebra: A Modern Introduction* by David Poole – are relevant:

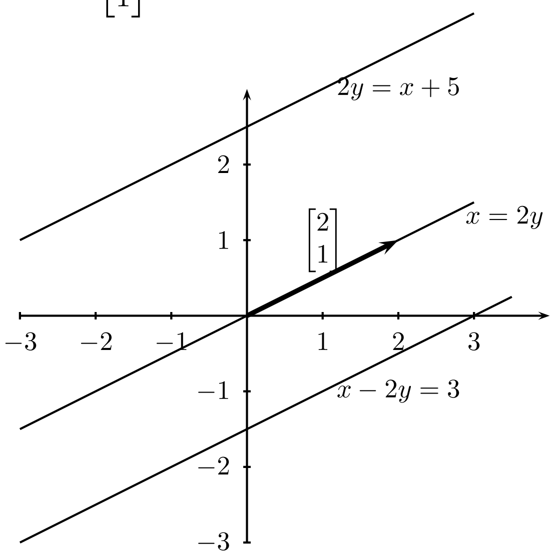
Exercises 1.3: 3, 5, 7, 11, 13, 15, 19, 21, 23.

Solutions

1. (a) (i) $\begin{bmatrix}1\\2\end{bmatrix}$ (ii) $\begin{bmatrix}2\\1\end{bmatrix}$ (iii) $\begin{bmatrix}1\\-2\end{bmatrix}$ (iv) $\begin{bmatrix}-2\\1\end{bmatrix}$ (v) $\begin{bmatrix}-2\\2\end{bmatrix}$ (vi) $\begin{bmatrix}2\\4\end{bmatrix}$ (vii) $\begin{bmatrix}1\\-2\end{bmatrix}$ (viii) $\begin{bmatrix}6\\3\end{bmatrix}$ (ix) $\begin{bmatrix}-4\\2\end{bmatrix}$ (x) $\begin{bmatrix}-2\\-1\end{bmatrix}$ (xi) $\begin{bmatrix}1\\-2\end{bmatrix}$

Note: In each case, any multiple of the vector given is also a correct answer.

(b) The lines (ii), (viii) and (x) are parallel to $\begin{bmatrix}2\\1\end{bmatrix}$, since the vectors in (ii), (viii) and (x) are multiples of $\begin{bmatrix}2\\1\end{bmatrix}$. Note that the lines all have gradient 1/2.



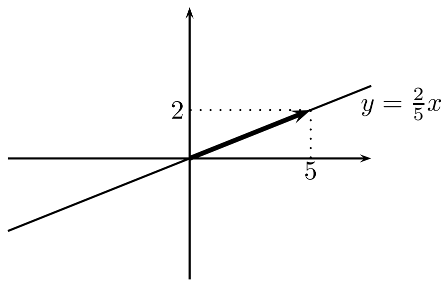
(c) The lines in (iii), (vii) and (xi) are perpendicular to $\begin{bmatrix}2\\1\end{bmatrix}$, since the dot product of the vectors in (iii), (vii) and (xi) with $\begin{bmatrix}2\\1\end{bmatrix}$ is zero. Note that the corresponding lines all have gradient -2.

2. (a) In $\mathbb{R}^3$, the equation $ax + by + cz = d$ (where a, b, c and d are real numbers) represents a plane. A normal to the plane $ax + by + cz = d$ is $\begin{bmatrix}a\\b\\c\end{bmatrix}$. Hence, a normal to the plane $x + y + 5z = 6$ is $\begin{bmatrix}1\\1\\5\end{bmatrix}$.

(b) Rewrite the equation as $3x - 8y + z = 7$. Then a normal is $\begin{bmatrix}3\\-8\\1\end{bmatrix}$.

(c) A normal to the plane $2x + y - z = 6$ is $\begin{bmatrix}2\\1\\-1\end{bmatrix}$. The length of this normal is $\sqrt{6}$, and so a unit normal is $\begin{bmatrix}\frac{2}{\sqrt{6}}\\\frac{1}{\sqrt{6}}\\-\frac{1}{\sqrt{6}}\end{bmatrix}$.

3. Multiplying the vector $\mathbf{v} = \begin{bmatrix}5\\2\end{bmatrix}$ by any real number t gives another vector in the same direction as $\mathbf{v}$, or in the opposite direction (if $t < 0$). Hence, all the vectors $\mathbf{x} = \begin{bmatrix}x\\y\end{bmatrix} = t\begin{bmatrix}5\\2\end{bmatrix}$, for $t \in \mathbb{R}$, form the straight line through $(0, 0)$ in the direction of $\begin{bmatrix}5\\2\end{bmatrix}$.

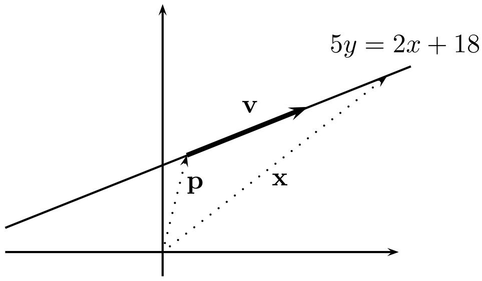


So the *vector equation* of the line is $\mathbf{x} = \begin{bmatrix}x\\y\end{bmatrix} = t\begin{bmatrix}5\\2\end{bmatrix} = \begin{bmatrix}5t\\2t\end{bmatrix}$, and any point (x, y) on the line is given by $x = 5t$, $y = 2t$ (where t is any real number). These are the *parametric equations* of the line.

The *Cartesian equation* is found by eliminating t from the parametric equations. We have $t = \dfrac{x}{5}$ and $t = \dfrac{y}{2}$, so $\dfrac{x}{5} = \dfrac{y}{2}$, or $5y = 2x$, or $y = \frac{2}{5}x$.

4. In this case, we find a vector equation for the line by starting at the point $P = (1, 4)$, which is the head of the position vector $\mathbf{p} = \overrightarrow{OP}$, and then adding any multiple of $\mathbf{v} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$. Then, by vector addition, the vector $\mathbf{x} = \mathbf{p} + t\,\mathbf{v}$, where t is any real number, is a vector whose head lies on the required straight line.

So the vector equation is:
$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} + t \begin{bmatrix} 5 \\ 2 \end{bmatrix}, \quad t \in \mathbb{R}.$
That is, $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1+5t \\ 4+2t \end{bmatrix}.$
Hence the parametric equations are:
$x = 1 + 5t,\ y = 4 + 2t, \quad t \in \mathbb{R}.$
Eliminating t from the parametric equations gives the Cartesian equation:
$t = \dfrac{x-1}{5}$, so $y = 4 + 2\left(\dfrac{x-1}{5}\right)$, or
$5y = 2x + 18.$



5. Using the same idea as in question 4, with $\mathbf{p} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$, the vector equation is

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1+2t \\ 2t \\ -1-t \end{bmatrix}, \quad t \in \mathbb{R}.$$

Hence, the parametric equations are

$$x = 1 + 2t, \quad y = 2t, \quad z = -1 - t, \quad t \in \mathbb{R}.$$

6. The line is in the direction of the vector $\overrightarrow{PQ} = \begin{bmatrix} -2 \\ 4 \\ -1 \end{bmatrix} - \begin{bmatrix} -4 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -6 \end{bmatrix}$, and it passes through the point P. So with $\mathbf{p} = \begin{bmatrix} -4 \\ 3 \\ 5 \end{bmatrix}$ and $\mathbf{v} = \overrightarrow{PQ}$, a vector equation is $\mathbf{x} = \mathbf{p} + t\,\mathbf{v}$.

$$\text{That is,} \quad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \\ 5 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ -6 \end{bmatrix} = \begin{bmatrix} -4+2t \\ 3+t \\ 5-6t \end{bmatrix} \quad t \in \mathbb{R}.$$

Hence the parametric equations are

$$x = -4 + 2t, \quad y = 3 + t, \quad z = 5 - 6t, \quad t \in \mathbb{R}.$$

7. The equation of a plane in normal form is $\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0$, where $\mathbf{n}$ is a vector normal to the plane, $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is an arbitrary vector in $\mathbb{R}^3$, and $\mathbf{p} = \overrightarrow{OP}$ is the position vector from the origin to the point P in the plane. (Note that the vector $\mathbf{x} - \mathbf{p}$ is an arbitrary vector in the plane, and we require it to be perpendicular to $\mathbf{n}$. That is, we require the dot product of $\mathbf{x} - \mathbf{p}$ and $\mathbf{n}$ to be zero.)

In this case, we have $\mathbf{n} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$ and $\mathbf{p} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$ and so the equation of the plane in normal form is

$$\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} \cdot \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} \right) = 0.$$

In order to find the Cartesian equation we simply calculate this dot product. We have:

$$\begin{aligned} \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} x - 2 \\ y - 3 \\ z - 5 \end{bmatrix} &= 0 \\ (x - 2) + 3(y - 3) - (z - 5) &= 0 \\ x + 3y - z &= 6 \end{aligned}$$

The Cartesian equation is $x + 3y - z = 6$.

8. The procedure for finding the equation of a plane given 3 points P, Q and R is as follows:

Find any two vectors in the plane - that is, find any two of $\overrightarrow{PQ}$, $\overrightarrow{PR}$ or $\overrightarrow{QR}$.

Then find the cross product of the two vectors that you have found. Remember that the cross product of two vectors is perpendicular to both of them, so the cross product of two vectors in a plane is perpendicular, or normal, to the plane.

Now the equation $\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0$ can be used, where $\mathbf{n}$ is the cross product, and any of the position vectors $\overrightarrow{OP}$, $\overrightarrow{OQ}$ or $\overrightarrow{OR}$ may be used for $\mathbf{p}$.

In this example, two vectors in the plane are $\overrightarrow{PQ} = \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \\ -6 \end{bmatrix}$ and

$$\overrightarrow{PR} = \begin{bmatrix} 4 \\ -4 \\ 4 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \\ 1 \end{bmatrix}.$$

A vector normal to the plane is therefore $\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{bmatrix} -40 \\ -16 \\ 24 \end{bmatrix}$.

The point P lies in the plane, and so we can take $\mathbf{p} = \overrightarrow{OP} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, and then an equation of the plane in normal form is

$$\begin{bmatrix} -40 \\ -16 \\ 24 \end{bmatrix} \cdot \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right) = 0.$$

Calculating the dot product in this equation we have

$$\begin{bmatrix} -40 \\ -16 \\ 24 \end{bmatrix} \cdot \begin{bmatrix} x-1 \\ y-2 \\ z-3 \end{bmatrix} = 0$$

$$-40(x-1) - 16(y-2) + 24(z-3) = 0.$$

The Cartesian form of the equation is therefore $-40x - 16y + 24z = 0$
or $\quad 5x + 2y - 3z = 0$.

9. If a line is perpendicular to a plane, then it is in the direction of a normal to the plane.

A normal to the plane $3x - 4y + z = 6$ is $\begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix}$ (see question 2).

So we want the line through the point $(1, 0, -2)$, in the direction of $\begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix}$. As in question 5, a vector equation for the line is

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + t \begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix}, \quad \in \mathbb{R}.$$

Parametric equations for the line therefore are

$$\left. \begin{aligned} x &= 1 + 3t \\ y &= -4t \\ z &= -2 + t \end{aligned} \right\} \quad t \in \mathbb{R}\ .$$

10. In vector form, the line has equation

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} + t \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \quad \in \mathbb{R}.$$

The direction of the line is therefore that of the vector $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$.

Now, if the line is parallel to the plane, then it will be perpendicular to a normal to the plane.

A normal to the plane is $\mathbf{n} = \begin{bmatrix} 4 \\ 4 \\ -5 \end{bmatrix}$.

Since two vectors are perpendicular if their dot product is zero, it is therefore sufficient to check that $\mathbf{v} \cdot \mathbf{n} = 0$:

$$\mathbf{v} \cdot \mathbf{n} \;=\; 2(4) + 3(4) + 4(-5) \;=\; 8 + 12 - 20 \;=\; 0\ .$$

Hence, $\mathbf{v}$ is perpendicular to $\mathbf{n}$, and the line is parallel to the plane.