The University of Sydney
School of Mathematics and Statistics

Tutorial 5 (Week 5)

MATH1014: Introduction to Linear Algebra — Semester 2, 2009

1. In each of the following draw a diagram to illustrate the system of linear equations. State whether the system is consistent or inconsistent. If the system is consistent, say how many solutions exist, and find them all.

(a)
$$\begin{aligned} x + y &= 6 \\ y &= 1 \end{aligned}$$

(b)
$$\begin{aligned} x + y &= 6 \\ 2x + 2y &= 1 \end{aligned}$$

(c)
$$\begin{aligned} x + y &= 6 \\ 2x + 2y &= 12 \end{aligned}$$

(d)
$$\begin{aligned} x + y &= 6 \\ 2x - 2y &= 0 \end{aligned}$$

2. Consider the equation $2x + y - 3z = 6$. How many variables are "free"? How many parameters are required for a solution? Write a parametric solution. Find three specific solutions.

Geometrically, what does the equation $2x + y - 3z = 6$ represent? What do your three specific solutions represent?

3. Consider the system of equations with augmented matrix $\left[\begin{array}{ccc|c} 2 & 1 & -3 & 6 \\ 0 & 1 & 1 & 4 \end{array}\right]$. How many variables are "free"? How many parameters are required for a solution? Write a parametric solution. Find two specific solutions.

Geometrically, what does the system of equations represent? What does your parametric solution represent?

4. Write down the augmented matrix corresponding to each of the following systems of equations.

(a)
$$\begin{aligned} 3x - 2y + z &= -4 \\ x + 4z &= 5 \end{aligned}$$

(b)
$$\begin{aligned} 3x - 2y &= -4 \\ 2x + 3y &= 5 \\ 5x + y &= 1 \end{aligned}$$

(c)
$$\begin{aligned} x - y + 5z &= 3 \\ 2x + y - 3z &= 0 \\ x + 4y + z &= 7 \end{aligned}$$

(d)
$$\begin{aligned} x_1 + 2x_2 \qquad - x_4 &= 1 \\ x_1 - x_2 + 3x_3 \qquad &= 2 \\ 2x_1 + 5x_2 - x_3 - x_4 &= 3 \end{aligned}$$

5. The following augmented matrices correspond to systems of equations in the three variables x, y and z. Use back substitution to find solutions to the systems.

(a) $\left[\begin{array}{ccc|c} 1 & 0 & 3 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$

(b) $\left[\begin{array}{ccc|c} 1 & -1 & 5 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$

(c) $\left[\begin{array}{ccc|c} 1 & 2 & -4 & 0 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 1 & 2 \end{array}\right]$

(d) $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$

6. Determine whether the systems of equations with the following augmented matrices are consistent or inconsistent. (Assume the variables are x and y in part (a), and x, y and z in parts (b) – (e).)

(a) $\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 5 & 0 \end{array}\right]$

(b) $\left[\begin{array}{ccc|c} 1 & 4 & 2 & 3 \\ 2 & 8 & 4 & 6 \\ 3 & 12 & 6 & 9 \end{array}\right]$

(c) $\left[\begin{array}{ccc|c} 1 & 4 & 2 & 3 \\ 1 & 4 & 2 & 5 \\ 2 & 8 & 4 & 6 \end{array}\right]$

(d) $\left[\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 2 & 5 & 7 & 0 \\ 2 & 8 & 4 & 0 \end{array}\right]$

(e) $\left[\begin{array}{ccc|c} 1 & 4 & 2 & 3 \\ 2 & 8 & 4 & 6 \\ 0 & 0 & 0 & 6 \end{array}\right]$

Further exercises

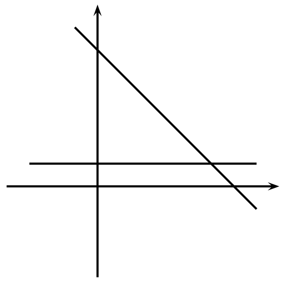
In addition to these exercises, the following exercises from the textbook – *Linear Algebra: A Modern Introduction* by David Poole – are relevant:

Exercises 2.1: 1, 3, 5, 11, 13, 15, 17, 19, 21, 23, 27, 29, 31.

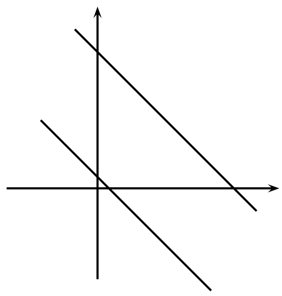
Solutions

1. A system of equations is *consistent* if it has at least one solution. A system of equations with no solutions is *inconsistent*.

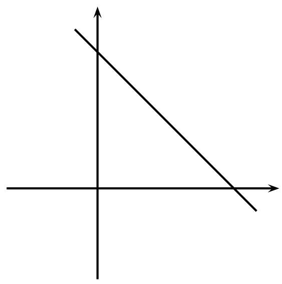
(a) The system is consistent, with a unique solution: $x = 5$, $y = 1$.
Geometrically, we have two lines meeting in the point $(5, 1)$.



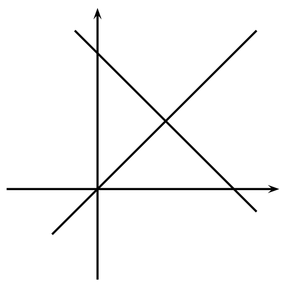
(b) The system is inconsistent, and there are no solutions. Note that the second equation is equivalent to $x + y = 0.5$. Clearly, we cannot have both $x + y = 6$ and $x + y = 0.5$.
Geometrically, we have two parallel lines, with no points in common.



(c) The system is consistent, and there are infinitely many solutions: $x = 6 - t$, $y = t$. Note that the second equation is equivalent to the first.
Geometrically, we have one straight line (or two lines that coincide). Every point on the line is a solution to the system.



(d) The system is consistent, with the unique solution: $x = 3$, $y = 3$.
As in part (a), we have two lines meeting in one point.



2. Two of the variables are free. (That is, we can assign any values we like to two of the variables, but then the third variable is determined.) Hence there are two parameters in the solution.
Choosing y and z as the parameters, and letting $y = s$ and $z = t$, we have $x = \dfrac{6 - s + 3t}{2}$. That is, the solution is

$$x = \frac{6 - s + 3t}{2}, \quad y = s, \quad z = t,$$

where s and t may take any real values. We can find specific solutions by choosing any values we like for s and t. Three examples are:

$$x = 3,\ y = 0,\ z = 0; \quad x = 4,\ y = 1,\ z = 1; \quad x = \frac{17}{2},\ y = -2,\ z = 3.$$

Geometrically, the equation $2x + y - 3z = 6$ represents a plane. The three specific solutions respresent the points $(3, 0, 0)$, $(4, 1, 1)$ and $(\frac{17}{2}, -2, 3)$, all of which lie in the plane.

3. The augmented matrix corresponds to the following system of equations:

$$\begin{aligned} 2x + y - 3z &= 6 \\ y + z &= 4 \end{aligned}$$

One variable is free, since we are free to assign any value we like to one of the variables, but then the other two variables are determined. So there is one parameter in the solution. Choosing z as the parameter and letting $z = t$, we see that $y = 4 - z = 4 - t$ and, by going back to the first equation and substituting for y and z in terms of t we find $2x = 6 - y + 3z = 6 - (4 - t) + 3t = 2 + 4t$, so that $x = 1 + 2t$. So a solution is

$$x = 1 + 2t, \quad y = 4 - t, \quad z = t \quad (t \in \mathbb{R}).$$

Two specific solutions are $x = 1$, $y = 4$, $z = 0$ and $x = 3$, $y = 3$, $z = 1$.

The system represents two planes. The parametric solution represents the straight line in which the two planes intersect.

4. (a) $\left[\begin{array}{ccc|c} 3 & -2 & 1 & -4 \\ 1 & 0 & 4 & 5 \end{array}\right]$

(b) $\left[\begin{array}{cc|c} 3 & -2 & -4 \\ 2 & 3 & 5 \\ 5 & 1 & 1 \end{array}\right]$

(c) $\left[\begin{array}{ccc|c} 1 & -1 & 5 & 3 \\ 2 & 1 & -3 & 0 \\ 1 & 4 & 1 & 7 \end{array}\right]$

(d) $\left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 1 \\ 1 & -1 & 3 & 0 & 2 \\ 2 & 5 & -1 & -1 & 3 \end{array}\right]$

5. (a) The corresponding system of equations is:

$$\begin{aligned} x + 3z &= 4 \\ y - z &= 2 \\ z &= 3 \end{aligned}$$

So $y = 2 + z = 2 + 3 = 5$, and $x = 4 - 3z = 4 - 9 = -5$.
That is, $x = -5$, $y = 5$, $z = 3$.

(b) The corresponding system of equations is:

$$\begin{aligned} x - y + 5z &= 0 \\ y + 3z &= 0 \\ z &= 0 \end{aligned}$$

By back substitution, beginning with $z = 0$, we have the solution $x = 0$, $y = 0$, $z = 0$.

(c) The corresponding system of equations is:

$$\begin{aligned} x + 2y - 4z &= 0 \\ y + 3z &= 7 \\ z &= 2 \end{aligned}$$

By back substitution, $y = 7 - 3z = 7 - 6 = 1$, and $x = 4z - 2y = 8 - 2 = 6$. That is, the solution is $x = 6$, $y = 1$, $z = 2$.

(d) The corresponding system of equations is:

$$\begin{aligned} x + 2y + 3z &= 4 \\ y - z &= 2 \end{aligned}$$

In this case, z is a free variable. Setting $z = t$ we have $y = 2 + t$ and $x = 4 - 2y - 3z = 4 - 2(2 + t) - 3t = -5t$. So a solution is

$$x = -5t, \quad y = 2 + t, \quad z = t \quad (t \in \mathbb{R}).$$

6. (a) The first two rows correspond to the equations

$$\begin{aligned} x + 2y &= 3 \\ 2x + 3y &= 5. \end{aligned}$$

Adding these two equations gives $3x + 5y = 8$, whereas the third row of the matrix corresponds to $3x + 5y = 0$. Hence the system is inconsistent.

(b) Note that row 2 $= 2\times$ row 1, and row 3 $= 3\times$ row 1. So the system corresponds to the single equation $x + 4y + 2z = 3$, and is consistent.

(c) Row 1 corresponds to the equation $x + 4y + 2z = 3$, while row 2 corresponds to the equation $x + 4y + 2z = 5$. The system is therefore inconsistent.

(d) It is clear that $x = y = z = 0$ is a solution. So the system is consistent. (When the right hand side of a system of equations consists of all zeros the equations are said to be *homogeneous*. Homogeneous equations are always consistent.)

(e) The last row corresponds to the equation $0 = 6$, which is clearly not true! The system is inconsistent.