The University of Sydney
School of Mathematics and Statistics

Tutorial 6 (Week 6)

MATH1014: Introduction to Linear Algebra — Semester 2, 2009

Preparatory questions (attempt before the tutorial)

1. Which of the following matrices are in row echelon form?

(a) $\begin{bmatrix}1&0&2\\0&0&0\\0&0&0\end{bmatrix}$ (e) $\begin{bmatrix}1&2&3\\0&0&1\\0&0&0\end{bmatrix}$ (i) $\begin{bmatrix}1&0&0&3&0\\0&0&1&0&0\end{bmatrix}$

(j) $\begin{bmatrix}1&2&3&4&5\\0&0&0&1&2\end{bmatrix}$

(b) $\begin{bmatrix}1&0&2\\0&1&0\\0&1&2\end{bmatrix}$ (f) $\begin{bmatrix}0&1&2\\0&0&1\\0&0&0\end{bmatrix}$ (k) $\begin{bmatrix}1&2\\0&0\\0&1\\0&0\end{bmatrix}$

(c) $\begin{bmatrix}1&2&3\\0&1&5\\0&0&0\end{bmatrix}$ (g) $\begin{bmatrix}1&2&3\\0&0&1\\0&0&1\end{bmatrix}$

(d) $\begin{bmatrix}1&2&3\\0&2&0\\0&0&3\end{bmatrix}$ (h) $\begin{bmatrix}1&0&0\\0&1&2\\0&0&1\end{bmatrix}$ (l) $\begin{bmatrix}1&2&3\\1&0&0\\0&1&1\\0&0&1\end{bmatrix}$

2. Reduce the following matrices to row echelon form.

(a) $\begin{bmatrix}4&3\\2&1\end{bmatrix}$ (b) $\begin{bmatrix}2&-4&-2&6\\3&1&6&6\end{bmatrix}$ (c) $\begin{bmatrix}-2&-4&7\\-3&-6&10\\1&2&-3\end{bmatrix}$

Tutorial exercises

3. Solve the following systems of equations by writing down the associated augmented matrix and row reducing.

(a) $\begin{aligned} x + y - z &= 0\\ 2x - y + z &= 9\\ x + z &= 10\end{aligned}$

(b) $\begin{aligned} -3x + 2y + z &= 4\\ 4x + y + 3z &= 9\\ x - y - z &= -4\end{aligned}$

(c) $\begin{aligned} x + 2y + 3z &= 0\\ 3x + 2y + z &= 0\end{aligned}$

(d) $\begin{aligned} x - y - z &= 0\\ x - 2y + z &= 0\\ 2x + z &= 0\end{aligned}$

(e) $\begin{aligned} x + 2y + 7z &= 5\\ x + y + 4z &= 3\\ 2x + 3y + 11z &= 7\end{aligned}$

(f) $\begin{aligned} x + 2y + z - w &= 4\\ 2x + 4y - z + 4w &= -1\\ -x - 2y + 2z - 5w &= 5\end{aligned}$

4. Interpret the solutions to Question **3** (a), (c), (d) and (e) geometrically.

5. Solve the following systems of equations.

(a)
$$\begin{array}{rcrcrcl} x & - & y & + & z & = & 0 \\ -x & + & 3y & + & z & = & 5 \\ 3x & + & y & + & 7z & = & 2 \end{array}$$

(b)
$$\begin{array}{rcrcrcrcr} -x_1 & + & 3x_2 & - & 2x_3 & + & 4x_4 & = & 0 \\ 2x_1 & - & 6x_2 & + & x_3 & - & 2x_4 & = & -3 \\ x_1 & - & 3x_2 & + & 4x_3 & - & 8x_4 & = & 2 \end{array}$$

(c)
$$\begin{array}{rcrcrcrcr} a & + & b & + & c & + & d & = & 4 \\ a & + & 2b & + & 3c & + & 4d & = & 10 \\ a & + & 3b & + & 6c & + & 10d & = & 20 \\ a & + & 4b & + & 10c & + & 20d & = & 35 \end{array}$$

6. Find the values of k (if any) such that the following system has (i) no solution; (ii) a unique solution; (iii) infinitely many solutions.

$$\begin{array}{rcrcrcr} x & - & 2y & + & 3z & = & 2 \\ x & + & y & + & z & = & k \\ 2x & - & y & + & 4z & = & k^2 \end{array}$$

7. Find the values of k (if any) such that the following system (i) is inconsistent; (ii) has infinitely many solutions; (iii) has a unique solution:

$$\begin{array}{rcrcrcr} x & & & - & 3z & = & -3 \\ -2x & - & ky & + & z & = & 2 \\ x & + & 2y & + & kz & = & 1 \end{array}$$

Further exercises

In addition to these exercises, the following exercises from the textbook – *Linear Algebra: A Modern Introduction* by David Poole – are relevant:

Exercises 2.2: 1, 3, 5, 7, 9, 11, 13, 25, 27, 29, 31, 33, 35, 37, 41, 43, 45.

Solutions

1. The matrices in (a), (c), (e), (f), (h), (i) and (j) are in row echelon form. The others are not.

2. (a) Perform the following elementary row operations, in order:

$$R1 \to R1 \div 4, \quad R2 \to R2 - 2R1, \quad R2 \to R2 \times -2$$

The resulting row echelon form is $\begin{bmatrix} 1 & 3/4 \\ 0 & 1 \end{bmatrix}$.

(b) Perform the following elementary row operations, in order:

$$R1 \to R1 \div 2, \quad R2 \to R2 - 3R1, \quad R2 \to R2 \div 7$$

The resulting row echelon form is $\begin{bmatrix} 1 & -2 & -1 & 3 \\ 0 & 1 & 9/7 & -3/7 \end{bmatrix}$.

(c) Perform the following elementary row operations, in order:

$$R1 \leftrightarrow R3, \quad R2 \to R2 + 3R1, \quad R3 \to R3 + 2R1, \quad R3 \to R3 - R2$$

The resulting row echelon form is $\begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

3. (a)

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 2 & -1 & 1 & 9 \\ 1 & 0 & 1 & 10 \end{array}\right] \xrightarrow[R3\to R3-R1]{R2\to R2-2R1} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -3 & 3 & 9 \\ 0 & -1 & 2 & 10 \end{array}\right]$$

$$\xrightarrow[R3\to R3+R2]{R2\to R2\div -3} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & 7 \end{array}\right]$$

By back substitution we find $z = 7$, $y = -3 + 7 = 4$, $x = 7 - 4 = 3$. The solution may also be written as $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 7 \end{bmatrix}$.

(b)

$$\left[\begin{array}{ccc|c} -3 & 2 & 1 & 4 \\ 4 & 1 & 3 & 9 \\ 1 & -1 & -1 & -4 \end{array}\right] \xrightarrow{R1 \leftrightarrow R3} \left[\begin{array}{ccc|c} 1 & -1 & -1 & -4 \\ 4 & 1 & 3 & 9 \\ -3 & 2 & 1 & 4 \end{array}\right]$$

$$\xrightarrow[R3 \to R3+3R1]{R2 \to R2-4R1} \left[\begin{array}{ccc|c} 1 & -1 & -1 & -4 \\ 0 & 5 & 7 & 25 \\ 0 & -1 & -2 & -8 \end{array}\right]$$

$$\xrightarrow[R3 \leftrightarrow R2]{R3 \to R3 \times -1} \left[\begin{array}{ccc|c} 1 & -1 & -1 & -4 \\ 0 & 1 & 2 & 8 \\ 0 & 5 & 7 & 25 \end{array}\right]$$

$$\xrightarrow[R3 \to R3 \div -3]{R3 \to R3-5R2} \left[\begin{array}{ccc|c} 1 & -1 & -1 & -4 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

By back substitution we find $z = 5,\ y = 8 - 2 \times 5 = -2$, $x = -4+5+(-2) = -1$. The solution may also be written as $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ 5 \end{bmatrix}$.

(c)

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 3 & 2 & 1 & 0 \end{array}\right] \xrightarrow{R2 \to R2-3R1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 0 \end{array}\right]$$

$$\xrightarrow{R2 \to R2 \div -4} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$$

z is a free variable. Let $z = t$, and by back substitution $y = -2t$ and $x = -3t - 2\times(-2t) = t$. The solution may also be written as $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

(d)

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 1 & -2 & 1 & 0 \\ 2 & 0 & 1 & 0 \end{array}\right] \xrightarrow[R3 \to R3-2R1]{R2 \to R2-R1} \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & 3 & 0 \end{array}\right]$$

$$\xrightarrow[R3 \to R3-2R2]{R2 \to R2 \times -1} \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 7 & 0 \end{array}\right]$$

The trivial solution $x = y = z = 0$ is the only solution. The solution may also be written as $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

(e)

$$\left[\begin{array}{ccc|c} 1 & 2 & 7 & 5 \\ 1 & 1 & 4 & 3 \\ 2 & 3 & 11 & 7 \end{array}\right] \xrightarrow[R3\to R3-2R1]{R2\to R2-R1} \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & -1 & -3 & -2 \\ 0 & -1 & -3 & -3 \end{array}\right]$$

$$\xrightarrow[R3\to R3+R2]{R2\to R2\times -1} \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

The system is inconsistent. There are no solutions.

(f)

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & -1 & 4 \\ 2 & 4 & -1 & 4 & -1 \\ -1 & -2 & 2 & -5 & 5 \end{array}\right] \xrightarrow[R3\to R3+R1]{R2\to R2-2R1} \left[\begin{array}{cccc|c} 1 & 2 & 1 & -1 & 4 \\ 0 & 0 & -3 & 6 & -9 \\ 0 & 0 & 3 & -6 & 9 \end{array}\right]$$

$$\xrightarrow[R2\to R2\div -3]{R3\to R3+R2} \left[\begin{array}{cccc|c} 1 & 2 & 1 & -1 & 4 \\ 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The variables y and w are free. Let $y = s$ and $w = t$, and then $z = 3 + 2t$, $x = 1 - 2s - t$.

The solution may also be written as $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$.

4. In **3** (a), the three equations represent three planes in space. Since there is a unique solution, the three planes meet in one point.

In **3** (c), the two equations represent two planes in space (both passing through the origin). The two planes meet in the straight line (through the origin) with parametric equations $x = t$, $y = -2t$, $z = t$.

The equations in **3** (d) represent 3 planes meeting in one point (the origin).

The equations in **3** (e) represent 3 planes such that there are no points in common to all three. Since no two of the planes are parallel (how do we know that?) the planes must meet in 3 separate straight lines.

5. (a) The augmented matrix reduces to $\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 2 & 2 & 5 \\ 0 & 0 & 0 & -8 \end{array}\right]$.

So the system is inconsistent and there are no solutions.

(b) The augmented matrix reduces to $\left[\begin{array}{cccc|c} 1 & -3 & 2 & -4 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$.

There are two free variables.
Let $x_2 = s$, $x_4 = t$ and then $x_3 = 1 + 2t$ and $x_1 = -2 + 3s$.

That is, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}$.

(c) The augmented matrix reduces to $\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 2 & 3 & 6 \\ 0 & 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$.

There is a unique solution: $a = b = c = d = 1$.

6.

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 1 & 1 & 1 & k \\ 2 & -1 & 4 & k^2 \end{array}\right] \xrightarrow[R3 \to R3 - 2R1]{R2 \to R2 - R1} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 0 & 3 & -2 & k-2 \\ 0 & 3 & -2 & k^2 - 4 \end{array}\right]$$

$$\xrightarrow[R3 \to R3 - 2R1]{R2 \to R2 - R1} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 0 & 3 & -2 & k-2 \\ 0 & 0 & 0 & k^2 - k - 2 \end{array}\right]$$

(i) The system has no solutions if $k^2 - k - 2 = (k-2)(k+1) \neq 0$; that is, for all values of k other than 2 or -1.

(ii) There are no values of k for which the solution is unique.

(iii) There are infinitely many solutions if $k = 2$ or -1.

7.

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -3 \\ -2 & -k & 1 & 2 \\ 1 & 2 & k & 1 \end{array}\right] \xrightarrow[R3 \to R3 - R1]{R2 \to R2 + 2R1} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 2 \\ 0 & -k & -5 & -4 \\ 0 & 2 & k+3 & 4 \end{array}\right]$$

$$\xrightarrow[R2 \leftrightarrow R3]{R3 \to R3 \div 2} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 2 \\ 0 & 1 & \frac{k+3}{2} & 2 \\ 0 & -k & -5 & -4 \end{array}\right]$$

$$\xrightarrow[R3 \to R3 \times 2]{R3 \to R3 + kR2} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 2 \\ 0 & 1 & \frac{k+3}{2} & 2 \\ 0 & 0 & k^2 + 3k - 10 & 4k - 8 \end{array}\right]$$

(i) The system is inconsistent if $k^2 + 3k - 10 = (k+5)(k-2) = 0$ and $4k - 8 \neq 0$. That is, if $k = -5$.

(ii) For infinitely many solutions we require $k^2 + 3k - 10 = (k+5)(k-2) = 0$ and $4k - 8 = 0$. That is, $k = 2$.

(iii) For a unique solution we require $k^2 + 3k - 10 = (k+5)(k-2) \neq 0$. That is, k can be any value other than 2 or -5.