THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Tutorial 7 (Week 7)

MATH1014: Introduction to Linear Algebra — Semester 2, 2009

Preparatory questions (attempt before the tutorial)

1. Let $A = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 4 & 0 \\ -1 & -2 & 6 \end{bmatrix}$, $C = \begin{bmatrix} -3 & 5 & -1 \end{bmatrix}$, $D = \begin{bmatrix} 3 & 4 \\ 2 & 0 \\ 0 & -7 \\ 1 & -3 \end{bmatrix}$.

Write down the size of each matrix.

2. Let $M = [m_{ij}] = \begin{bmatrix} 6 & 0 & 3 & -5 \\ 0 & 7 & 2 & 4 \\ 1 & 3 & -2 & 0 \end{bmatrix}$. Write down

(a) m_{22} (b) m_{33} (c) m_{14} (d) m_{32} (e) m_{34}.

Tutorial exercises

3. A coffee merchant sells three blends of coffee. A bag of the house blend has 300 grams of Columbian beans and 200 grams of French roast beans. A bag of the special blend has 200 grams of Columbian beans, 200 grams of Kenyan beans and 100 grams of French roast beans. A bag of the gourmet blend has 100 grams of Columbian beans, 200 grams of Kenyan beans and 200 grams of French roast beans. The merchant has 30 kg of Columbian beans, 15 kg of Kenyan beans and 25 kg of French roast beans, and wishes to use them all up. Determine whether or not he can, and if so, how many bags of each blend he should produce.

4. Now assume that the house blend has 300 grams of Columbian beans, 50 grams of Kenyan beans and 150 grams of French roast beans, the special blend has 200 grams of Columbian, 200 grams of Kenyan and 100 grams of French roast and the gourmet blend has 100 grams of Columbian, 350 grams of Kenyan and 50 grams of French roast. The merchant has 30 kg of Columbian beans, 15 kg of Kenyan and 15 kg of French roast available. The respective profits per bag are \$0.50, \$1.50 and \$2.00 for the house, special and gourmet blends.

The merchant wants to package *all* the beans *and* maximise his profit. How many bags of each type should he produce, and what is the resulting profit?

5. Balance the chemical equation for each of the following reactions:

(a) $CO_2 + H_2O \longrightarrow C_6H_{12}O_6 + O_2$

(b) $HClO_4 + P_4O_{10} \longrightarrow H_3PO_4 + Cl_2O_7$

6. The sum of the ages of Annie, Bert and Chris is 60. Annie is older than Bert by the same number of years as Bert is older than Chris. When Bert is as old as Annie is now, Annie will be three times as old as Chris is now. What are their ages?

7. Let $A = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 4 \\ -1 & -2 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 2 \\ 4 & -3 \end{bmatrix}$, $D = \begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix}$.

Find each of the following.

(a) $A + B$	(e) $2A$	(i) AB	(m) $A(BC)$	(q) B^2
(b) $A - B$	(f) $-B$	(j) BA	(n) $(AB)C$	(r) A^2B^2
(c) $B - C$	(g) $3C$	(k) CD	(o) $ABCD$	
(d) $D + C$	(h) $\frac{1}{5}D$	(l) BC	(p) A^2	

8. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & -1 \\ -1 & 2 \end{bmatrix}$, $C = \begin{bmatrix} 3 & 4 & 2 \end{bmatrix}$, $D = \begin{bmatrix} -1 \\ -1 \\ 5 \end{bmatrix}$.

Find any of the following that exist:

AB, AC, AD, BA, BC, BD, CA, CB, CD, DA, DB, DC.

9. Find a 2×2 matrix M such that $M^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ but every entry of M is nonzero.

Further exercises

In addition to these exercises, the following exercises from the textbook – *Linear Algebra: A Modern Introduction* by David Poole – should be attempted:

Exercises 2.4: 1, 3, 7, 9, 11, 31, 33, 39.

Exercises 3.1: 1, 3, 5, 7, 9, 11, 15.

Solutions

1. A matrix with m rows and n columns is an $m \times n$ matrix. Hence, A is a 2×2 matrix, B is 2×3, C is 1×3 and D is 4×2.

2. The entry m_{ij} is the entry in row i column j. Hence we have
(a) $m_{22} = 7$ (b) $m_{33} = -2$ (c) $m_{14} = -5$ (d) $m_{32} = 3$ (e) $m_{33} = 0$.

3. Let x be the number of bags of house blend, y the number of bags of special blend and z the number of bags of gourmet blend that the merchant should make in order to use up all his coffee beans.

Then the number of grams of Colombian beans used will be

$$300x + 200y + 100z,$$

the number of grams of Kenyan beans used will be

$$200y + 200z,$$

and the number of grams of French roast beans used will be

$$200x + 100y + 200z.$$

Hence, if the merchant wishes to use 30 kg of Colombian, 15 kg of Kenyan and 25 kg of French roast beans, then we must have

$$\begin{aligned} 300x + 200y + 100z &= 30000 \\ 200y + 200z &= 15000 \\ 200x + 100y + 200z &= 25000 \end{aligned}$$

So long as this system of equations is consistent, the merchant will be able to use all the beans.

The system of equations is equivalent to

$$\begin{aligned} 3x + 2y + 1z &= 300 \\ y + z &= 75 \\ 2x + y + 2z &= 250 \end{aligned}$$

with augmented matrix $\left[\begin{array}{ccc|c} 3 & 2 & 1 & 300 \\ 0 & 1 & 1 & 75 \\ 2 & 1 & 2 & 250 \end{array}\right]$. This matrix reduces to $\left[\begin{array}{ccc|c} 1 & \frac{2}{3} & \frac{1}{3} & 100 \\ 0 & 1 & 1 & 75 \\ 0 & 0 & 1 & 45 \end{array}\right]$ and so by back substitution we obtain $x = 65$, $y = 30$, $z = 45$.

Hence, the merchant should make 65 bags of house blend, 30 bags of special blend and 45 bags of gourmet blend in order to use up all the beans.

4. In this case, with x, y and z as in Question **3**, we have

$$\begin{aligned} 300x + 200y + 100z &= 30000 \\ 50x + 200y + 350z &= 15000 \\ 150x + 100y + 50z &= 15000 \end{aligned}$$

or

$$\begin{aligned} 3x + 2y + z &= 300 \\ x + 4y + 7z &= 300 \\ 3x + 2y + z &= 300 \end{aligned}$$

The augmented matrix for this system reduces to $\left[\begin{array}{ccc|c} 1 & 4 & 7 & 300 \\ 0 & 1 & 2 & 60 \\ 0 & 0 & 0 & 0 \end{array}\right]$. So z is a free variable, and with $z = t$ we have $y = 60 - 2t$ and $x = 60 + t$. Since we cannot have negative numbers of bags of coffee, we have to ensure that $60 + t \geq 0$ and $60 - 2t \geq 0$ and $t \geq 0$. That is, we must have $0 \leq t \leq 30$.

Now, if the merchant sells $60 + t$ bags of house blend, $60 - 2t$ bags of special blend and t bags of gourmet blend, the profit (in dollars) will be

$$0.5(60 + t) + 1.5(60 - 2t) + 2t = 120 - 0.5t.$$

Given the constraints on t ($0 \leq t \leq 30$) the profit will be a maximum when $t = 0$. Hence, the merchant should make 60 bags of house blend, 60 bags of special blend and no bags of gourmet blend. The profit will then be \$120.

5. (a) We need to find a, b, c and d such that

$$a\,CO_2 + b\,H_2O \longrightarrow c\,C_6H_{12}O_6 + d\,O_2$$

is a balanced equation. That is, we require

$$\begin{aligned} a &= 6c && \text{(carbon)} \\ 2a + b &= 6c + 2d && \text{(oxygen)} \\ 2b &= 12c && \text{(hydrogen)} \end{aligned}$$

or, rewriting:

$$\begin{array}{ccccccccc} a & & & - & 6c & & & = & 0 \\ 2a & + & b & - & 6c & - & 2d & = & 0 \\ & & 2b & - & 12c & & & = & 0 \end{array}$$

The augmented matrix for this system is $\left[\begin{array}{cccc|c} 1 & 0 & -6 & 0 & 0 \\ 2 & 1 & -6 & -2 & 0 \\ 0 & 2 & -12 & 0 & 0 \end{array}\right]$, and it reduces to $\left[\begin{array}{cccc|c} 1 & 0 & -6 & 0 & 0 \\ 0 & 1 & 6 & -2 & 0 \\ 0 & 0 & 1 & -\frac{1}{6} & 0 \end{array}\right]$.

So there are infinitely many solutions of the form $a = t$, $b = t$, $c = \frac{t}{6}$, $d = t$. The smallest positive value of t which gives integer values for a, b, c and d is $t = 6$. The balanced equation is

$$6\,CO_2 + 6\,H_2O \longrightarrow C_6H_{12}O_6 + 6\,O_2.$$

(b) We need to find a, b, c and d such that

$$a\,\mathrm{HClO_4} + b\,\mathrm{P_4O_{10}} \longrightarrow c\,\mathrm{H_3PO_4} + d\,\mathrm{Cl_2O_7}$$

is a balanced equation. That is, we require

$$\begin{aligned} a &= 3c && \text{(hydrogen)} \\ a &= 2d && \text{(chlorine)} \\ 4a + 10b &= 4c + 7d && \text{(oxygen)} \\ 4b &= c && \text{(phosphorus)} \end{aligned}$$

or, rewriting:

$$\begin{array}{rcrcrcrcl} a & & & - & 3c & & & = & 0 \\ a & & & & & - & 2d & = & 0 \\ 4a & + & 10b & - & 4c & - & 7d & = & 0 \\ & & 4b & - & c & & & = & 0 \end{array}$$

The augmented matrix for this system is $\left[\begin{array}{cccc|c} 1 & 0 & -3 & 0 & 0 \\ 1 & 0 & 0 & -2 & 0 \\ 4 & 10 & -4 & -7 & 0 \\ 0 & 4 & -1 & 0 & 0 \end{array}\right]$, and it reduces to $\left[\begin{array}{cccc|c} 1 & 0 & -3 & 0 & 0 \\ 0 & 1 & -\frac{1}{4} & 0 & 0 \\ 0 & 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$.

So there are infinitely many solutions of the form $a = 2t$, $b = \frac{t}{6}$, $c = \frac{2t}{3}$, $d = t$. The smallest positive value of t which gives integer values for a, b, c and d is $t = 6$. The balanced equation is

$$12\,\mathrm{HClO_4} + \mathrm{P_4O_{10}} \longrightarrow 4\,\mathrm{H_3PO_4} + 6\,\mathrm{Cl_2O_7}.$$

6. Let the ages be a (Annie), b (Bert) and c (Chris). Then we have:

$$\begin{aligned} a + b + c &= 60 \\ a - b &= b - c \quad \text{or} \quad a - 2b + c = 0 \\ a + (a - b) &= 3c \end{aligned}$$

The augmented matrix reduces to $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 60 \\ 0 & 1 & 0 & 20 \\ 0 & 0 & 1 & 12 \end{array}\right]$.

Hence, Chris is 12, Bert is 20 and Annie is 28.

7. (a) $A+B=\begin{bmatrix}4 & 2\\ -2 & 1\end{bmatrix}$

(b) $A-B=\begin{bmatrix}-2 & -6\\ 0 & 5\end{bmatrix}$

(c) $B-C=\begin{bmatrix}3 & 2\\ -5 & 1\end{bmatrix}$

(d) $D+C=\begin{bmatrix}10 & 2\\ 4 & 2\end{bmatrix}$

(e) $2A=\begin{bmatrix}2 & -4\\ -2 & 6\end{bmatrix}$

(f) $-B=\begin{bmatrix}-3 & -4\\ 1 & 2\end{bmatrix}$

(g) $3C=\begin{bmatrix}0 & 6\\ 12 & -9\end{bmatrix}$

(h) $\frac{1}{5}D=\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}$

(i) $AB=\begin{bmatrix}5 & 8\\ -6 & -10\end{bmatrix}$

(j) $BA\begin{bmatrix}-1 & 6\\ 1 & -4\end{bmatrix}$

(k) $CD=\begin{bmatrix}0 & 10\\ 40 & -15\end{bmatrix}$

(l) $BC=\begin{bmatrix}16 & -6\\ -8 & 4\end{bmatrix}$

(m) $A(BC)=\begin{bmatrix}32 & -14\\ -40 & 18\end{bmatrix}$

(n) $(AB)C=\begin{bmatrix}32 & -14\\ -40 & 18\end{bmatrix}$

(o) $ABCD=\begin{bmatrix}320 & -70\\ -400 & 90\end{bmatrix}$

(p) $A^2=\begin{bmatrix}3 & -8\\ -4 & 11\end{bmatrix}$

(q) $B^2=\begin{bmatrix}5 & 4\\ -1 & 0\end{bmatrix}$

(r) $A^2B^2=\begin{bmatrix}23 & 12\\ -31 & -16\end{bmatrix}$

8. $AB=\begin{bmatrix}-1 & 5\\ 1 & 3\end{bmatrix}$, $AD=\begin{bmatrix}12\\ 0\end{bmatrix}$, $BA=\begin{bmatrix}3 & 2 & 1\\ -2 & 0 & 2\\ 5 & 2 & -1\end{bmatrix}$, $CB=\begin{bmatrix}2 & 3\end{bmatrix}$,

$CD=[3]$, $DC=\begin{bmatrix}-3 & -4 & -2\\ -3 & -4 & -2\\ 15 & 20 & 10\end{bmatrix}$.

The other products are not defined.

9. One such matrix is $\begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix}$.