The University of Sydney
School of Mathematics and Statistics

Tutorial 8 (Week 8)

MATH1014: Introduction to Linear Algebra — Semester 2, 2009

Preparatory questions (attempt before the tutorial)

1. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $ad - bc \neq 0$, then $A^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

(a) Use this formula to find the inverse of each of the following matrices, if it exists:

$$A = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 5 & 4 \\ 3 & 3 \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}.$$

(b) Verify that: $AA^{-1} = I_2$, $B^{-1}B = I_2$, $DD^{-1} = I_2$, $E^{-1}E = I_2$.
(I_2 is the 2×2 identity matrix, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.)

Tutorial exercises

2. Find the inverse of $\begin{bmatrix} 2 & 3 & 0 \\ 1 & -2 & -1 \\ 2 & 0 & -1 \end{bmatrix}$.

3. Find the inverse of $\begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 2 \\ 2 & 3 & -1 \end{bmatrix}$.

4. Let $A = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$.

(a) Find AB, A^{-1}, and B^{-1}.

(b) Find $B^{-1}A^{-1}$. Calculate the products $(AB)(B^{-1}A^{-1})$ and $(B^{-1}A^{-1})(AB)$. What do these products tell you about $(AB)^{-1}$?

5. Use B^{-1} from Question **4** to solve

(a)
$$\begin{aligned} x + 2y + 3z &= 0 \\ 2x + 3y + z &= 0 \\ 3x + y + 2z &= 0 \end{aligned}$$

(b)
$$\begin{aligned} x + 2y + 3z &= 2 \\ 2x + 3y + z &= 2 \\ 3x + y + 2z &= 4 \end{aligned}$$

6. Let $A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 & -1 \\ -3 & 5 & 3 \end{bmatrix}$.

(a) Find AB and BA.

(b) Does A have an inverse? Does B have an inverse?

7. A square matrix in which all the entries off the main diagonal are zero is called a *diagonal matrix*.

(a) Let A be the diagonal matrix $\begin{bmatrix} 5 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$. Find A^{-1}.

(b) The general $n \times n$ diagonal matrix is:

$$D = \begin{bmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{bmatrix}$$

Explain why D is not invertible if $d_i = 0$ for some $i = 1, 2, \dots, n$.
When all the diagonal entries are non-zero, what is D^{-1}?

Further exercises

In addition to these exercises, the following exercises from the textbook – *Linear Algebra: A Modern Introduction* by David Poole – should be attempted:
Exercises 3.3: 1, 3, 5, 7, 9, 11, 49, 51, 55.

Solutions

1. (a) $A^{-1} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$, $B^{-1} = \dfrac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, C^{-1} does not exist, $D^{-1} = \dfrac{1}{3}\begin{bmatrix} 3 & -4 \\ -3 & 5 \end{bmatrix}$,

$E^{-1} = \dfrac{1}{2}\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$.

2.

$$\left[\begin{array}{ccc|ccc} 2 & 3 & 0 & 1 & 0 & 0 \\ 1 & -2 & -1 & 0 & 1 & 0 \\ 2 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R1 \leftrightarrow R2} \left[\begin{array}{ccc|ccc} 1 & -2 & -1 & 0 & 1 & 0 \\ 2 & 3 & 0 & 1 & 0 & 0 \\ 2 & 0 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R3 \to R3 - 2R1]{R2 \to R2 - 2R1} \left[\begin{array}{ccc|ccc} 1 & -2 & -1 & 0 & 1 & 0 \\ 0 & 7 & 2 & 1 & -2 & 0 \\ 0 & 4 & 1 & 0 & -2 & 1 \end{array}\right]$$

$$\xrightarrow{\substack{R2 \to R2/7 \\ R1 \to R1 + 2R2 \\ R3 \to R3 - 4R2}} \left[\begin{array}{ccc|ccc} 1 & 0 & -3/7 & 2/7 & 3/7 & 0 \\ 0 & 1 & 2/7 & 1/7 & -2/7 & 0 \\ 0 & 0 & -1/7 & -4/7 & -6/7 & 1 \end{array}\right]$$

$$\xrightarrow{\substack{R3 \to R3 \times -7 \\ R1 \to R1 + \frac{3}{7}R3 \\ R2 \to R2 - \frac{2}{7}R3}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 3 & -3 \\ 0 & 1 & 0 & -1 & -2 & 2 \\ 0 & 0 & 1 & 4 & 6 & -7 \end{array}\right]$$

Hence, the inverse of $\begin{bmatrix} 2 & 3 & 0 \\ 1 & -2 & -1 \\ 2 & 0 & -1 \end{bmatrix}$ is $\begin{bmatrix} 2 & 3 & -3 \\ -1 & -2 & 2 \\ 4 & 6 & -7 \end{bmatrix}$.

(Multiply the matrix and its inverse together to verify that the product is the 3×3 identity matrix.)

3.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 3 & 1 & 2 & 0 & 1 & 0 \\ 2 & 3 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R3 \to R3 - 2R1]{R2 \to R2 - 3R1} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 4 & -4 & -3 & 1 & 0 \\ 0 & 5 & -5 & -2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R3 \to R3 - 5R2]{R2 \to R2/4} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -3/4 & 1/4 & 0 \\ 0 & 0 & 0 & 7/4 & -5/4 & 1 \end{array}\right]$$

The matrix cannot be reduced to the identity matrix, and hence it does not have an inverse.

4. (a) $AB = \begin{bmatrix} -2 & -3 & -1 \\ 7 & 7 & 4 \\ 1 & 2 & 3 \end{bmatrix}$, $A^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 2 & 1 & 0 \end{bmatrix}$, $B^{-1} = \dfrac{1}{18}\begin{bmatrix} -5 & 1 & 7 \\ 1 & 7 & -5 \\ 7 & -5 & 1 \end{bmatrix}$.

(A^{-1} and B^{-1} are found by the same method as used in Question **2**.)

(b) $B^{-1}A^{-1} = \dfrac{1}{18}\begin{bmatrix} 13 & 7 & -5 \\ -17 & -5 & 1 \\ 7 & 1 & 7 \end{bmatrix}$, $\quad (AB)(B^{-1}A^{-1}) = (B^{-1}A^{-1})(AB) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

The products verify that $(AB)^{-1} = B^{-1}A^{-1}$.

5. (a) The system of equations is $B\mathbf{x} = \mathbf{0}$, where B is the matrix in Question **4**, $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. This is a homogeneous system, and so we know that one solution is the trivial solution $x = y = z = 0$. The fact that B^{-1} exists tells us that the trivial solution is the only solution, since we have

$$B^{-1}B\mathbf{x} = \mathbf{x} = B^{-1}\mathbf{0} = \frac{1}{18}\begin{bmatrix} -5 & 1 & 7 \\ 1 & 7 & -5 \\ 7 & -5 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

(b) Here we have $B\mathbf{x} = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$ and so

$$\mathbf{x} = B^{-1}\begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix} = \frac{1}{18}\begin{bmatrix} -5 & 1 & 7 \\ 1 & 7 & -5 \\ 7 & -5 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 20 \\ -4 \\ 8 \end{bmatrix}.$$

That is, $x = \dfrac{10}{9}$, $y = -\dfrac{2}{9}$, $z = \dfrac{4}{9}$.

6. (a) $AB = \begin{bmatrix} -4 & 10 & 5 \\ 3 & -5 & -3 \\ -10 & 20 & 11 \end{bmatrix}$, $\quad BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(b) Neither matrix has an inverse. Only square matrices have inverses.

7. (a) The matrix A will be reduced to I_3 (the 3×3 identity matrix) if $R1 \to R1\div5$, $R2 \to R2 \times -1$ and $R3 \to R3 \div -2$. Performing these row operations on I_3 gives $A^{-1} = \begin{bmatrix} 1/5 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1/2 \end{bmatrix}$.

(b) If $d_i = 0$ for any i, D will contain a row of zeros, and will not be able to be reduced to I_n.

If each d_i is non-zero, then D can be reduced to I_n by the row operations $Ri \to Ri \div d_i$ for each i.

That is, D is invertible if and only if $d_i \neq 0$ for $i = 1, 2, \ldots, n$.

When $d_i \neq 0$ for $i = 1, 2, \ldots, n$, $\quad D^{-1} = \begin{bmatrix} \frac{1}{d_1} & 0 & \ldots & 0 \\ 0 & \frac{1}{d_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \frac{1}{d_n} \end{bmatrix}$