THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Tutorial 9 (Week 9)

MATH1014: Introduction to Linear Algebra — Semester 2, 2009

Preparatory questions (attempt before the tutorial)

1. Which of the following are stochastic matrices?

(a) $\begin{bmatrix} 0.2 & 0.9 \\ 0.8 & 0.1 \end{bmatrix}$ (b) $\begin{bmatrix} 0.2 & 0.8 \\ 0.9 & 0.1 \end{bmatrix}$ (c) $\begin{bmatrix} \frac{1}{12} & \frac{1}{9} & \frac{1}{6} \\ \frac{1}{2} & 0 & \frac{5}{6} \\ \frac{5}{12} & \frac{8}{9} & 0 \end{bmatrix}$ (d) $\begin{bmatrix} -1 & \frac{1}{3} & \frac{1}{2} \\ 0 & \frac{1}{3} & \frac{1}{2} \\ 2 & \frac{1}{3} & 0 \end{bmatrix}$

2. Let $P = \begin{bmatrix} 0.4 & 0.3 \\ 0.6 & 0.7 \end{bmatrix}$ be the transition matrix for a Markov chain with 2 states. What is the probability that

(a) something in state 1 remains in state 1?

(b) something in state 1 moves to state 2?

(c) something in state 2 remains in state 2?

(d) something in state 2 moves to state 1?

3. Let $L = \begin{bmatrix} 0 & 1 & 3 \\ 0.6 & 0 & 0 \\ 0 & 0.7 & 0 \end{bmatrix}$ be the Leslie matrix for an animal population with 3 age groups.

(a) On average, how many female offspring do females in each of the age groups produce?

(b) What percentage of the female population in the first age group survives to the second age group?

(c) What percentage of the female population in the second age group survives to the third age group?

Tutorial exercises

4. Let $P = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{3} & 0 \end{bmatrix}$ be the transition matrix for a Markov chain with 3 states.

Let $\mathbf{x}_0 = \begin{bmatrix} 120 \\ 180 \\ 90 \end{bmatrix}$ be the initial state vector.

(a) Calculate $\mathbf{x}_1$ and $\mathbf{x}_2$.

(b) How many of the initial state 1 population will still be in state 1 after 2 steps?

(c) How many of the initial state 2 population will be in state 3 after 2 steps?

(d) Find the steady state vector.

5. Based on accumulated data concerning the heights of male children relative to their fathers, it has been determined that the probabilities that a tall man will have a tall, medium-height or short child are 0.6, 0.2 and 0.2 respectively. The probabilities that a man of medium height will have a tall, medium-height or short child are 0.1, 0.7 and 0.2 respectively, and the probabilities that a short man will have a tall, medium-height or short child are 0.2, 0.4 and 0.4 respectively.

(a) Write down the transition matrix for this Markov chain.

(b) What is the probability that a short man will have a tall grandson?

(c) If 20% of the current male population is tall, 50% is of medium height and 30% is short, what will the distribution be in three generations?

(d) According to this model, what will the distribution be in the long run?

6. An insect population with 4 age classes has Leslie matrix $L = \begin{bmatrix} 0 & 1 & 2 & 5 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0.7 & 0 & 0 \\ 0 & 0 & 0.3 & 0 \end{bmatrix}$.

If the initial population vector is $\mathbf{x}_0 = \begin{bmatrix} 10 \\ 10 \\ 10 \\ 10 \end{bmatrix}$, find $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$.

7. By observing a certain colony of mice, researchers found that all animals die within 3 years. Of those offspring that are females, 60% live for at least one year. Of these, 20% reach their second birthday. The females who are under 1 year of age have, on average, 3 female offspring. Those females between 1 and 2 years of age have, on average, one female offspring. None of the females of age 2 give birth.

(a) Construct the Leslie matrix for this situation.

(b) Suppose that the current population distribution for females is $\begin{bmatrix} 100 \\ 60 \\ 30 \end{bmatrix}$. Find the population vector for the next year.

8. Suppose the Leslie matrix for an animal population is $L = \begin{bmatrix} 0 & 0 & 20 \\ 0.1 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}$, and

$$\mathbf{x}_0 = \begin{bmatrix} 10 \\ 10 \\ 10 \end{bmatrix}.$$

Find $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$, and hence determine the future behaviour of this population.

Further exercises

In addition to these exercises, the following exercises from the textbook – *Linear Algebra: A Modern Introduction* by David Poole – should be attempted:

Exercises 3.7: 1, 2, 3, 4, 9, 11, 19.

Answers to selected exercises

1. (a) and (c) are stochastic matrices, (b) and (d) are not.

2. (a) 0.4 (b) 0.6 (c) 0.7 (d) 0.3

3. (a) First age group: 0, second age group: 1, third age group: 3.

(b) 0.6 (c) 0.7

4. (a) $\mathbf{x}_1 = \begin{bmatrix} 150 \\ 120 \\ 120 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 155 \\ 120 \\ 115 \end{bmatrix}.$

(b) 50 (c) 50 (d) $\begin{bmatrix} 156 \\ 117 \\ 117 \end{bmatrix}$

5. (a) $\begin{bmatrix} 0.6 & 0.1 & 0.2 \\ 0.2 & 0.7 & 0.4 \\ 0.2 & 0.2 & 0.4 \end{bmatrix}$ (b) 0.24 (c) $\mathbf{x}_3 = \begin{bmatrix} 0.2457 \\ 0.5039 \\ 0.2504 \end{bmatrix}$ (d) $\begin{bmatrix} 0.25 \\ 0.5 \\ 0.25 \end{bmatrix}$

6. $\mathbf{x}_1 = \begin{bmatrix} 80 \\ 5 \\ 7 \\ 3 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 34 \\ 40 \\ 3.5 \\ 2.1 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 57.5 \\ 17 \\ 28 \\ 1.05 \end{bmatrix}.$

7. (a) $\begin{bmatrix} 3 & 1 & 0 \\ 0.6 & 0 & 0 \\ 0 & 0.2 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 360 \\ 60 \\ 12 \end{bmatrix}$

8. $\mathbf{x}_1 = \begin{bmatrix} 200 \\ 1 \\ 5 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 100 \\ 20 \\ 0.5 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 10 \\ 10 \\ 10 \end{bmatrix}.$