

Tutorial 9 (Week 9)

**Preparatory questions
(attempt before the tutorial)**

1. Which of the following are stochastic matrices?

(a) $\begin{bmatrix} 0.2 & 0.9 \\ 0.8 & 0.1 \end{bmatrix}$ (b) $\begin{bmatrix} 0.2 & 0.8 \\ 0.9 & 0.1 \end{bmatrix}$ (c) $\begin{bmatrix} \frac{1}{12} & \frac{1}{9} & \frac{1}{6} \\ \frac{1}{2} & 0 & \frac{5}{6} \\ \frac{5}{12} & \frac{8}{9} & 0 \end{bmatrix}$ (d) $\begin{bmatrix} -1 & \frac{1}{3} & \frac{1}{2} \\ 0 & \frac{1}{3} & \frac{1}{2} \\ 2 & \frac{1}{3} & 0 \end{bmatrix}$

2. Let $P = \begin{bmatrix} 0.4 & 0.3 \\ 0.6 & 0.7 \end{bmatrix}$ be the transition matrix for a Markov chain with 2 states.

What is the probability that

- (a) something in state 1 remains in state 1?
- (b) something in state 1 moves to state 2?
- (c) something in state 2 remains in state 2?
- (d) something in state 2 moves to state 1?

3. Let $L = \begin{bmatrix} 0 & 1 & 3 \\ 0.6 & 0 & 0 \\ 0 & 0.7 & 0 \end{bmatrix}$ be the Leslie matrix for an animal population with 3 age groups.

- (a) On average, how many female offspring do females in each of the age groups produce?
- (b) What percentage of the female population in the first age group survives to the second age group?
- (c) What percentage of the female population in the second age group survives to the third age group?

Tutorial exercises

4. Let $P = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{3} & 0 \end{bmatrix}$ be the transition matrix for a Markov chain with 3 states.

Let $\mathbf{x}_0 = \begin{bmatrix} 120 \\ 180 \\ 90 \end{bmatrix}$ be the initial state vector.

- (a) Calculate \mathbf{x}_1 and \mathbf{x}_2 .

- (b) How many of the initial state 1 population will still be in state 1 after 2 steps?
- (c) How many of the initial state 2 population will be in state 3 after 2 steps?
- (d) Find the steady state vector.

5. Based on accumulated data concerning the heights of male children relative to their fathers, it has been determined that the probabilities that a tall man will have a tall, medium-height or short child are 0.6, 0.2 and 0.2 respectively. The probabilities that a man of medium height will have a tall, medium-height or short child are 0.1, 0.7 and 0.2 respectively, and the probabilities that a short man will have a tall, medium-height or short child are 0.2, 0.4 and 0.4 respectively.

- (a) Write down the transition matrix for this Markov chain.
- (b) What is the probability that a short man will have a tall grandson?
- (c) If 20% of the current male population is tall, 50% is of medium height and 30% is short, what will the distribution be in three generations?
- (d) According to this model, what will the distribution be in the long run?

6. An insect population with 4 age classes has Leslie matrix $L = \begin{bmatrix} 0 & 1 & 2 & 5 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0.7 & 0 & 0 \\ 0 & 0 & 0.3 & 0 \end{bmatrix}$.

If the initial population vector is $\mathbf{x}_0 = \begin{bmatrix} 10 \\ 10 \\ 10 \\ 10 \end{bmatrix}$, find \mathbf{x}_1 , \mathbf{x}_2 and \mathbf{x}_3 .

7. By observing a certain colony of mice, researchers found that all animals die within 3 years. Of those offspring that are females, 60% live for at least one year. Of these, 20% reach their second birthday. The females who are under 1 year of age have, on average, 3 female offspring. Those females between 1 and 2 years of age have, on average, one female offspring. None of the females of age 2 give birth.

- (a) Construct the Leslie matrix for this situation.

(b) Suppose that the current population distribution for females is $\begin{bmatrix} 100 \\ 60 \\ 30 \end{bmatrix}$. Find the population vector for the next year.

8. Suppose the Leslie matrix for an animal population is $L = \begin{bmatrix} 0 & 0 & 20 \\ 0.1 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}$, and

$$\mathbf{x}_0 = \begin{bmatrix} 10 \\ 10 \\ 10 \end{bmatrix}.$$

Find \mathbf{x}_1 , \mathbf{x}_2 and \mathbf{x}_3 , and hence determine the future behaviour of this population.

Further exercises

In addition to these exercises, the following exercises from the textbook – *Linear Algebra: A Modern Introduction* by David Poole – should be attempted:

Exercises 3.7: 1, 2, 3, 4, 9, 11, 19.

Solutions

1. A matrix is a stochastic matrix if all entries are non-negative, and the entries in each column add to 1. Hence, (a) and (c) are stochastic matrices, (b) and (d) are not.
2. (a) 0.4 (b) 0.6 (c) 0.7 (d) 0.3
3. (a) Females in the first age group produce no female offspring; those in the second age group produce 1 on average; those in the third age group produce 3 on average.
(b) 0.6 (c) 0.7

$$4. \quad (a) \quad \mathbf{x}_1 = P \mathbf{x}_0 = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{3} & 0 \end{bmatrix} \begin{bmatrix} 120 \\ 180 \\ 90 \end{bmatrix} = \begin{bmatrix} 150 \\ 120 \\ 120 \end{bmatrix},$$

$$\mathbf{x}_2 = P \mathbf{x}_1 = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{3} & 0 \end{bmatrix} \begin{bmatrix} 150 \\ 120 \\ 120 \end{bmatrix} = \begin{bmatrix} 155 \\ 120 \\ 115 \end{bmatrix}.$$

- (b) The probability that something in state 1 will still be in state 1 after 2 steps is the (1, 1) entry of P^2 .

$$\text{Now, } P^2 = \begin{bmatrix} \frac{5}{12} & \frac{7}{18} & \frac{7}{18} \\ \frac{1}{3} & \frac{1}{3} & \frac{2}{9} \\ \frac{1}{4} & \frac{5}{18} & \frac{7}{18} \end{bmatrix}, \text{ and the (1, 1) entry is } \frac{5}{12}.$$

Initially, the state 1 population numbered 120.

So after two steps, $\frac{5}{12} \times 120 = 50$ of those will still be in state 1.

- (c) The probability of moving from state 2 to state 3 after 2 steps is the (3, 2) entry of P^2 or $\frac{5}{18}$. Initially, the state 2 population numbered 180, and so after two steps, $\frac{5}{18} \times 180 = 50$ of those will have moved to state 3.
- (d) The steady state vector is a vector \mathbf{x} such that $P\mathbf{x} = \mathbf{x}$, or $(P - I)\mathbf{x} = \mathbf{0}$.

$$\text{That is, } \begin{bmatrix} -\frac{1}{2} & \frac{1}{3} & \frac{1}{3} \\ 0 & -\frac{2}{3} & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{3} & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The augmented matrix for this system reduces to

$$\left[\begin{array}{ccc|c} 1 & -\frac{2}{3} & -\frac{2}{3} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right],$$

so we have $\mathbf{x}_3 = t$, $\mathbf{x}_2 = t$ and $\mathbf{x}_1 = \frac{4t}{3}$. Now, the total population is $120 + 180 + 90 = 390$, so we require $\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 = \frac{10t}{3} = 390$, or $t = 117$.

Hence, the steady state vector is $\begin{bmatrix} 156 \\ 117 \\ 117 \end{bmatrix}$.

5. (a) We have 3 states in this question – being tall, of medium height, or short. If we take being tall as state 1, being of medium height as state 2 and being short as state 3, the transition matrix is

$$P = \begin{bmatrix} 0.6 & 0.1 & 0.2 \\ 0.2 & 0.7 & 0.4 \\ 0.2 & 0.2 & 0.4 \end{bmatrix}.$$

- (b) The probability that a short man will have a tall grandson is the probability of moving from state 3 to state 1 in 2 steps. This is given by the (1, 3) entry of P^2 , and is equal to the dot product of row 1 of P with column 3 of P – that is, 0.24.

- (c) Taking $\mathbf{x}_0 = \begin{bmatrix} 0.2 \\ 0.5 \\ 0.3 \end{bmatrix}$ we calculate $\mathbf{x}_1 = P \mathbf{x}_0 = \begin{bmatrix} 0.23 \\ 0.51 \\ 0.26 \end{bmatrix}$,
 $\mathbf{x}_2 = P \mathbf{x}_1 = \begin{bmatrix} 0.241 \\ 0.507 \\ 0.252 \end{bmatrix}$ and $\mathbf{x}_3 = P \mathbf{x}_2 = \begin{bmatrix} 0.2457 \\ 0.5039 \\ 0.2504 \end{bmatrix}$.

So after three generations, approximately 25% of the male population will be tall, approximately 50% will be of medium height and approximately 25% will be short.

- (d) As in Question 4 (d), we solve $(P - I) \mathbf{x} = \mathbf{0}$. In this case the augmented matrix for the system is

$$\left[\begin{array}{ccc|c} -0.4 & 0.1 & 0.2 & 0 \\ 0.2 & -0.3 & 0.4 & 0 \\ 0.2 & 0.2 & -0.6 & 0 \end{array} \right],$$

and it reduces to

$$\left[\begin{array}{ccc|c} 1 & 1 & -3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right].$$

Hence, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ 2t \\ t \end{bmatrix}$, and in this case we want the steady state prob-

ability vector. That is, we want $x_1 + x_2 + x_3 = 4t = 1$. So $t = \frac{1}{4}$ and the

steady state probability vector is $\begin{bmatrix} 0.25 \\ 0.5 \\ 0.25 \end{bmatrix}$.

That is, in the long run, 25% of the male population is tall, 50% is of medium height and 25% is short.

$$6. \quad \mathbf{x}_1 = L \mathbf{x}_0 = \begin{bmatrix} 80 \\ 5 \\ 7 \\ 3 \end{bmatrix}, \quad \mathbf{x}_2 = L \mathbf{x}_1 = \begin{bmatrix} 34 \\ 40 \\ 3.5 \\ 2.1 \end{bmatrix}, \quad \mathbf{x}_3 = L \mathbf{x}_2 = \begin{bmatrix} 57.5 \\ 17 \\ 28 \\ 1.05 \end{bmatrix}.$$

$$7. \quad (a) \quad \text{The Leslie matrix, } L = \begin{bmatrix} 3 & 1 & 0 \\ 0.6 & 0 & 0 \\ 0 & 0.2 & 0 \end{bmatrix}.$$

$$(b) \quad \text{The population vector for the next year is } L \times \begin{bmatrix} 100 \\ 60 \\ 30 \end{bmatrix} = \begin{bmatrix} 360 \\ 60 \\ 12 \end{bmatrix}.$$

$$8. \quad \mathbf{x}_1 = L \mathbf{x}_0 = \begin{bmatrix} 200 \\ 1 \\ 5 \end{bmatrix}, \quad \mathbf{x}_2 = L \mathbf{x}_1 = \begin{bmatrix} 100 \\ 20 \\ 0.5 \end{bmatrix}, \quad \mathbf{x}_3 = L \mathbf{x}_2 = \begin{bmatrix} 10 \\ 10 \\ 10 \end{bmatrix}.$$

Since $\mathbf{x}_3 = \mathbf{x}_0$ the population distribution has a cyclic nature, repeating every 3 years.