THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Tutorial 10 (Week 10)

MATH1014: Introduction to Linear Algebra — Semester 2, 2009

1. In each of the following, show that $\mathbf{v}$ is an eigenvector of A and find the corresponding eigenvalue.

(a) $A = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(b) $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$.

(c) $A = \begin{bmatrix} -1 & 1 \\ 6 & 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

(d) $A = \begin{bmatrix} 4 & -2 \\ 5 & -7 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$.

(e) $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & -2 \\ 1 & 0 & 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$.

(f) $A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$.

2. Find the eigenvalues of each of the following matrices.

(a) $\begin{bmatrix} 0 & 4 \\ -1 & 5 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix}$

Solutions

1. (a) $A\mathbf{v} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3\,\mathbf{v}$, so $\mathbf{v}$ is an eigenvector. The corresponding eigenvalue is 3.

(b) $A\mathbf{v} = \begin{bmatrix} -3 \\ 3 \end{bmatrix} = -\,\mathbf{v}$, so $\mathbf{v}$ is an eigenvector. The corresponding eigenvalue is -1.

(c) $A\mathbf{v} = \begin{bmatrix} -3 \\ 6 \end{bmatrix} = -3\,\mathbf{v}$, so $\mathbf{v}$ is an eigenvector. The corresponding eigenvalue is -3.

(d) $A\mathbf{v} = \begin{bmatrix} 12 \\ 6 \end{bmatrix} = 3\,\mathbf{v}$, so $\mathbf{v}$ is an eigenvector. The corresponding eigenvalue is 3.

(e) $A\mathbf{v} = \begin{bmatrix} 6 \\ -3 \\ 3 \end{bmatrix} = 3\,\mathbf{v}$, so $\mathbf{v}$ is an eigenvector. The corresponding eigenvalue is 3.

(f) $A\mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0\,\mathbf{v}$, and $\mathbf{v} \neq 0$, so it is an eigenvector. The corresponding eigenvalue is 0.

2. To find the eigenvalues, λ, of a matrix A we solve the equation $\det(A - \lambda I) = |A - \lambda I| = 0$.

(a) Let $A = \begin{bmatrix} 0 & 4 \\ -1 & 5 \end{bmatrix}$, and then

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} -\lambda & 4 \\ -1 & 5 - \lambda \end{vmatrix} \\ &= -\lambda(5 - \lambda) + 4 \\ &= \lambda^2 - 5\lambda + 4 \\ &= (\lambda - 4)(\lambda - 1) \end{aligned}$$

So $|A - \lambda I| = 0$ when $\lambda = 4$ or 1. That is, the eigenvalues are 4 and 1.

(b) Let $A = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}$, and then

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} 2 - \lambda & 2 \\ 2 & -1 - \lambda \end{vmatrix} \\ &= (2 - \lambda)(-1 - \lambda) - 4 \\ &= \lambda^2 - \lambda - 6 \\ &= (\lambda - 3)(\lambda + 2) \end{aligned}$$

So $|A - \lambda I| = 0$ when $\lambda = 3$ or -2. That is, the eigenvalues are 3 and -2.

(c) Let $A = \begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix}$, and then

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} 1-\lambda & 4 \\ 4 & 1-\lambda \end{vmatrix} \\ &= (1-\lambda)^2 - 16 \\ &= \lambda^2 - 2\lambda - 15 \\ &= (\lambda - 5)(\lambda + 3) \end{aligned}$$

So $|A - \lambda I| = 0$ when $\lambda = 5$ or -3. That is, the eigenvalues are 5 and -3.