

Tutorial 11 (Week 11)

**Preparatory questions
(attempt before the tutorial)**

1. Find the eigenvalues, and all the corresponding eigenvectors, for each of the following matrices.

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$

Tutorial exercises

2. Compute each of the following determinants in two ways; first by expansion using the first row, and then by expansion using the first column.

(a) $\begin{vmatrix} 0 & 1 & -1 \\ 2 & 3 & -2 \\ -1 & 3 & 0 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix}$ (c) $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$

3. Compute the following determinants using expansion by whichever row or column seems most convenient.

(a) $\begin{vmatrix} 5 & 2 & 2 \\ -1 & 1 & 2 \\ 3 & 0 & 0 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & 1 & -1 \\ 2 & 0 & 1 \\ 3 & -2 & 1 \end{vmatrix}$ (c) $\begin{vmatrix} -4 & 1 & 3 \\ 2 & -2 & 4 \\ 1 & -1 & 0 \end{vmatrix}$

4. Write down the determinants of the following matrices:

(a) $\begin{bmatrix} 5 & 0 & 0 \\ 3 & -2 & 0 \\ 1 & -5 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 3 & 8 \\ 0 & -6 & -7 \\ 0 & 0 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} -4 & -5 & 11 \\ 0 & 0 & 0 \\ 2 & -1 & 2 \end{bmatrix}$
(d) $\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -2 \\ 1 & 0 & 0 \end{bmatrix}$ (e) $\begin{bmatrix} 0 & 0 & 5 \\ 6 & 0 & 0 \\ 0 & -3 & 0 \end{bmatrix}$ (f) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ -5 & 0 & 8 \end{bmatrix}$

5. Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ and suppose $\det(A) = 10$. Use the properties of determinants to find each of the following.

(a) $\det(2A)$ (b) $\det(-A)$ (c) $\begin{vmatrix} 2a & 2b & 2c \\ g & h & i \\ d & e & f \end{vmatrix}$

$$(d) \begin{vmatrix} a & b & c \\ 5d+a & 5e+b & 5f+c \\ g-2a & h-2b & i-2c \end{vmatrix} \quad (e) \begin{vmatrix} 3a & 3b & 3c \\ -d & -e & -f \\ 6g & 6h & 6i \end{vmatrix}$$

6. Let $A = \begin{bmatrix} 2k & 5 \\ k & k-3 \end{bmatrix}$. Find any values of k for which A is not invertible.

7. Let $B = \begin{bmatrix} k & k & 0 \\ k^2 & 2 & k \\ 0 & k & k \end{bmatrix}$. Find all values of k for which B is invertible.

8. Use expansion by the first row to show that

$$\begin{vmatrix} a+3d & b+3e & c+3f \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} 3d & 3e & 3f \\ d & e & f \\ g & h & i \end{vmatrix}.$$

Now show by direct evaluation (without using any properties of determinants) that

$$\begin{vmatrix} 3d & 3e & 3f \\ d & e & f \\ g & h & i \end{vmatrix} = 0, \text{ and hence that } \begin{vmatrix} a+3d & b+3e & c+3f \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}.$$

Further exercises

In addition to these exercises, the following exercises from the textbook – *Linear Algebra: A Modern Introduction* by David Poole – should be attempted:

Exercises 4.2: 26 – 31, 35 – 40.

Answers to selected exercises

1. (a) Eigenvalues are 1 and 2.

Eigenvectors corresponding to the eigenvalue 1: $\begin{bmatrix} t \\ 0 \end{bmatrix}$, $t \in \mathbb{R}$, $t \neq 0$

Eigenvectors corresponding to the eigenvalue 2: $\begin{bmatrix} 0 \\ t \end{bmatrix}$, $t \in \mathbb{R}$, $t \neq 0$.

- (b) Eigenvalues are ± 1 .

Eigenvectors corresponding to the eigenvalue 1: $\begin{bmatrix} -t \\ t \end{bmatrix}$, $t \in \mathbb{R}$, $t \neq 0$

Eigenvectors corresponding to the eigenvalue -1 : $\begin{bmatrix} t \\ t \end{bmatrix}$, $t \in \mathbb{R}$, $t \neq 0$.

- (c) Eigenvalues are 2 and -3 .

Eigenvectors corresponding to the eigenvalue 2: $\begin{bmatrix} t \\ t \end{bmatrix}$, $t \in \mathbb{R}$, $t \neq 0$

Eigenvectors corresponding to the eigenvalue -1 : $\begin{bmatrix} -\frac{3t}{2} \\ t \end{bmatrix}$, $t \in \mathbb{R}$, $t \neq 0$.

- (d) Only one eigenvalue, 0. Eigenvectors: $\begin{bmatrix} t \\ t \end{bmatrix}$, $t \in \mathbb{R}$, $t \neq 0$

2. (a) -7 (b) -2 (c) 0

3. (a) 6 (b) 7 (c) -12

4. (a) 10 (b) -36 (c) 0 (d) 2 (e) -90 (f) 0

5. (a) $2^3 \times 10 = 80$ (b) -10 (c) -20 (d) 50 (e) -180

6. $k = 0$ or $\frac{11}{2}$.

7. Any value of k other than $0, 1$ or -2 .