

Tutorial 11 (Week 11)

**Preparatory questions
(attempt before the tutorial)**

1. Find the eigenvalues, and all the corresponding eigenvectors, for each of the following matrices.

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$

Tutorial exercises

2. Compute each of the following determinants in two ways; first by expansion using the first row, and then by expansion using the first column.

(a) $\begin{vmatrix} 0 & 1 & -1 \\ 2 & 3 & -2 \\ -1 & 3 & 0 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix}$ (c) $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$

3. Compute the following determinants using expansion by whichever row or column seems most convenient.

(a) $\begin{vmatrix} 5 & 2 & 2 \\ -1 & 1 & 2 \\ 3 & 0 & 0 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & 1 & -1 \\ 2 & 0 & 1 \\ 3 & -2 & 1 \end{vmatrix}$ (c) $\begin{vmatrix} -4 & 1 & 3 \\ 2 & -2 & 4 \\ 1 & -1 & 0 \end{vmatrix}$

4. Write down the determinants of the following matrices:

(a) $\begin{bmatrix} 5 & 0 & 0 \\ 3 & -2 & 0 \\ 1 & -5 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 3 & 8 \\ 0 & -6 & -7 \\ 0 & 0 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} -4 & -5 & 11 \\ 0 & 0 & 0 \\ 2 & -1 & 2 \end{bmatrix}$
(d) $\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -2 \\ 1 & 0 & 0 \end{bmatrix}$ (e) $\begin{bmatrix} 0 & 0 & 5 \\ 6 & 0 & 0 \\ 0 & -3 & 0 \end{bmatrix}$ (f) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ -5 & 0 & 8 \end{bmatrix}$

5. Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ and suppose $\det(A) = 10$. Use the properties of determinants to find each of the following.

(a) $\det(2A)$ (b) $\det(-A)$ (c) $\begin{vmatrix} 2a & 2b & 2c \\ g & h & i \\ d & e & f \end{vmatrix}$

$$(d) \begin{vmatrix} a & b & c \\ 5d+a & 5e+b & 5f+c \\ g-2a & h-2b & i-2c \end{vmatrix} \quad (e) \begin{vmatrix} 3a & 3b & 3c \\ -d & -e & -f \\ 6g & 6h & 6i \end{vmatrix}$$

6. Let $A = \begin{bmatrix} 2k & 5 \\ k & k-3 \end{bmatrix}$. Find any values of k for which A is not invertible.

7. Let $B = \begin{bmatrix} k & k & 0 \\ k^2 & 2 & k \\ 0 & k & k \end{bmatrix}$. Find all values of k for which B is invertible.

8. Use expansion by the first row to show that

$$\begin{vmatrix} a+3d & b+3e & c+3f \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} 3d & 3e & 3f \\ d & e & f \\ g & h & i \end{vmatrix}.$$

Now show by direct evaluation (without using any properties of determinants) that

$$\begin{vmatrix} 3d & 3e & 3f \\ d & e & f \\ g & h & i \end{vmatrix} = 0, \text{ and hence that } \begin{vmatrix} a+3d & b+3e & c+3f \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}.$$

Further exercises

In addition to these exercises, the following exercises from the textbook – *Linear Algebra: A Modern Introduction* by David Poole – should be attempted:

Exercises 4.2: 26 – 31, 35 – 40.

Solutions

1. Find the eigenvalues, λ , by solving $|A - \lambda I| = 0$, and then find the eigenvectors corresponding to a particular eigenvalue λ by solving the system of equations $(A - \lambda I)\mathbf{x} = \mathbf{0}$.

- (a) Since the matrix is diagonal, the eigenvalues are the entries on the diagonal. That is, the eigenvalues are 1 and 2.

When $\lambda = 1$ we solve $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{x} = \mathbf{0}$.

The eigenvectors are all vectors of the form $\begin{bmatrix} t \\ 0 \end{bmatrix}$, $t \in \mathbb{R}$, $t \neq 0$.

When $\lambda = 2$ we solve $\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \mathbf{x} = \mathbf{0}$.

The eigenvectors are all vectors of the form $\begin{bmatrix} 0 \\ t \end{bmatrix}$, $t \in \mathbb{R}$, $t \neq 0$.

- (b) $\begin{vmatrix} -\lambda & -1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = 0$ when $\lambda = \pm 1$. So the eigenvalues are 1 and -1 .

When $\lambda = 1$ we solve $\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \mathbf{x} = \mathbf{0}$.

The augmented matrix reduces to $\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right]$.

The eigenvectors are all vectors of the form $\begin{bmatrix} -t \\ t \end{bmatrix}$, $t \in \mathbb{R}$, $t \neq 0$.

When $\lambda = -1$ we solve $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \mathbf{x} = \mathbf{0}$.

The augmented matrix reduces to $\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array} \right]$.

The eigenvectors are all vectors of the form $\begin{bmatrix} t \\ t \end{bmatrix}$, $t \in \mathbb{R}$, $t \neq 0$.

- (c) $\begin{vmatrix} -1 - \lambda & 3 \\ 2 & -\lambda \end{vmatrix} = \lambda^2 + \lambda - 6 = (\lambda - 2)(\lambda + 3) = 0$ when $\lambda = 2$ or -3 .

So the eigenvalues are 2 and -3 .

When $\lambda = 2$ we solve $\begin{bmatrix} -3 & 3 \\ 2 & -2 \end{bmatrix} \mathbf{x} = \mathbf{0}$.

The augmented matrix reduces to $\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array} \right]$.

The eigenvectors are all vectors of the form $\begin{bmatrix} t \\ t \end{bmatrix}$, $t \in \mathbb{R}$, $t \neq 0$.

When $\lambda = -3$ we solve $\begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix} \mathbf{x} = \mathbf{0}$.

The augmented matrix reduces to $\left[\begin{array}{cc|c} 1 & 3/2 & 0 \\ 0 & 0 & 0 \end{array} \right]$.

The eigenvectors are all vectors of the form $\begin{bmatrix} -\frac{3t}{2} \\ t \end{bmatrix}$, $t \in \mathbb{R}$, $t \neq 0$.

(d) $\begin{vmatrix} 1-\lambda & -1 \\ 1 & -1-\lambda \end{vmatrix} = \lambda^2 = 0$ when $\lambda = 0$.

So the only eigenvalue is 0, and we solve $\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \mathbf{x} = \mathbf{0}$.

The eigenvectors are all vectors of the form $\begin{bmatrix} t \\ t \end{bmatrix}$, $t \in \mathbb{R}$, $t \neq 0$.

2. (a) Expansion by the first row:

$$-\begin{vmatrix} 2 & -2 \\ -1 & 0 \end{vmatrix} - \begin{vmatrix} 2 & 3 \\ -1 & 3 \end{vmatrix} = -7$$

Expansion by the first column:

$$-2 \begin{vmatrix} 1 & -1 \\ 3 & 0 \end{vmatrix} - \begin{vmatrix} 1 & -1 \\ 3 & -2 \end{vmatrix} = -7$$

(b) Expansion by the first row:

$$-\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = -2$$

Expansion by the first column:

$$\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} - \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = -2$$

(c) Expansion by the first row:

$$\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} = -3 + 12 - 9 = 0$$

Expansion by the first column:

$$\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 4 \begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix} + 7 \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} = -3 + 24 - 21 = 0$$

3. (a) Expansion by row 3: $3 \begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} = 6$

(b) Expansion by row 2:

$$-2 \begin{vmatrix} 1 & -1 \\ -2 & 1 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix} = 7$$

(c) Expansion by column 3:

$$3 \begin{vmatrix} 2 & -2 \\ 1 & -1 \end{vmatrix} - 4 \begin{vmatrix} -4 & 1 \\ 1 & -1 \end{vmatrix} = -12$$

4. (a) Since the matrix is triangular, its determinant is equal to the product of the diagonal entries: $5 \times -2 \times -1 = 10$

- (b) Again, the determinant is the product of the diagonal entries:
 $3 \times -6 \times 2 = -36$
- (c) Using expansion by row 2, we have the determinant $= 0$.
- (d) Using expansion by column 1, the determinant $= 1 \times \begin{vmatrix} -1 & 0 \\ 0 & -2 \end{vmatrix} = 2$.
- (e) Using expansion by row 1, the determinant $= 5 \times \begin{vmatrix} 6 & 0 \\ 0 & -3 \end{vmatrix} = -90$.
- (f) Since row 2 $= 2 \times$ row 1, the determinant $= 0$.

5. (a) To obtain $2A$, each entry of A is multiplied by 2. That is, each row is multiplied by 2, and so $\det(2A) = 2^3 \times 10 = 80$.
- (b) To obtain $-A$, each entry of A is multiplied by -1 . That is, each row is multiplied by -1 , and so $\det(-A) = (-1)^3 \times 10 = -10$.
 (Note: In general, $\det(-A) \neq -\det A$. For example, if A is a 2×2 matrix, then $\det(-A) = \det A$.)
- (c) Row 1 of A has been multiplied by 2, and rows 2 and 3 have been swapped. Hence, the determinant $= 2 \times -1 \times 10 = -20$.
- (d) Here we have row 2 of A multiplied by 5, and then row 1 added to row 2, and twice row 1 subtracted from row 3. Only the first operation affects the value of the determinant, and so $\begin{vmatrix} a & b & c \\ 5d+a & 5e+b & 5f+c \\ g-2a & h-2b & i-2c \end{vmatrix} = 5 \times 10 = 50$.
- (e) Row 1 of A has been multiplied by 3, row 2 by -1 and row 3 by 6. Hence, the value of the determinant is $3 \times -1 \times 6 \times 10 = -180$.

6. We need any values of k for which $\det(A) = 0$.

Now, $\det(A) = 2k(k-3) - 5k = 2k^2 - 11k = k(2k-11) = 0$ if $k = 0$ or $\frac{11}{2}$.

So A is not invertible if $k = 0$ or $\frac{11}{2}$.

(Note that if $k = 0$ the matrix is $\begin{bmatrix} 0 & 5 \\ 0 & -3 \end{bmatrix}$, and if $k = \frac{11}{2}$ the matrix is $\begin{bmatrix} \frac{11}{2} & 5 \\ \frac{11}{2} & \frac{5}{2} \end{bmatrix}$.)

7. Here we require $\det(B) \neq 0$.

Expanding by the first column we have

$$\det(B) = k(2k - k^2) - k^2 \times k^2 = -k^4 - k^3 + 2k^2 = -k^2(k+2)(k-1).$$

So $\det(B) = 0$ if $k = 0, 1, -2$, and B is invertible for any value of k other than $0, 1$ or -2 .

8. Expanding by the first row we have

$$\begin{aligned}
& \begin{vmatrix} a+3d & b+3e & c+3f \\ d & e & f \\ g & h & i \end{vmatrix} \\
&= (a+3d) \begin{vmatrix} e & f \\ h & i \end{vmatrix} - (b+3e) \begin{vmatrix} d & f \\ g & i \end{vmatrix} + (c+3f) \begin{vmatrix} d & e \\ g & h \end{vmatrix} \\
&= \left(a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix} \right) + \left(3d \begin{vmatrix} e & f \\ h & i \end{vmatrix} - 3e \begin{vmatrix} d & f \\ g & i \end{vmatrix} + 3f \begin{vmatrix} d & e \\ g & h \end{vmatrix} \right) \\
&= \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} 3d & 3e & 3f \\ d & e & f \\ g & h & i \end{vmatrix}.
\end{aligned}$$

Now,

$$\begin{aligned}
\begin{vmatrix} 3d & 3e & 3f \\ d & e & f \\ g & h & i \end{vmatrix} &= 3d \begin{vmatrix} e & f \\ h & i \end{vmatrix} - 3e \begin{vmatrix} d & f \\ g & i \end{vmatrix} + 3f \begin{vmatrix} d & e \\ g & h \end{vmatrix} \\
&= 3dei - 3dfh - 3edi + 3efg + 3fdh - 3feg = 0
\end{aligned}$$

and hence $\begin{vmatrix} a+3d & b+3e & c+3f \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}.$