The University of Sydney
School of Mathematics and Statistics

Tutorial 12 (Week 12)

MATH1014: Introduction to Linear Algebra — Semester 2, 2009

Preparatory questions (attempt before the tutorial)

1. Show that $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -19 \\ -4 \\ 1 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ are eigenvectors of the matrix $A = \begin{bmatrix} 2 & 4 & -3 \\ 0 & 3 & 0 \\ -1 & 5 & 4 \end{bmatrix}$. What are the corresponding eigenvalues?

Tutorial exercises

2. Write down the eigenvalues of each of the following matrices.

(a) $\begin{bmatrix} 5 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 0 & 0 \\ 6 & 7 & 0 \\ 9 & 2 & 1 \end{bmatrix}$

3. Find the eigenvalues, and corresponding eigenvectors of each of the following matrices:

(a) $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ (b) $B = \begin{bmatrix} 1 & 0 & 2 \\ 3 & -1 & 3 \\ 2 & 0 & 1 \end{bmatrix}$

4. Let $P = \begin{bmatrix} 0.2 & 0.3 & 0.4 \\ 0.6 & 0.1 & 0.4 \\ 0.2 & 0.6 & 0.2 \end{bmatrix}$ be the transition matrix of a Markov process.

(a) Verify that P has an eigenvalue of 1.

(b) Show that $\mathbf{u} = \begin{bmatrix} 24 \\ 28 \\ 27 \end{bmatrix}$ is a corresponding eigenvector.

(c) Find the steady state probability vector.

5. Every Leslie matrix has a unique positive eigenvalue, and a corresponding eigenvector with positive entries. Find the unique positive eigenvalue, and a corresponding eigenvector, for each of the following Leslie matrices.

(a) $L = \begin{bmatrix} 0 & 3 \\ \frac{2}{3} & 0 \end{bmatrix}$ (b) $M = \begin{bmatrix} 0 & 4 & 4 \\ \frac{3}{4} & 0 & 0 \\ 0 & \frac{2}{3} & 0 \end{bmatrix}$

6. Consider an animal population whose growth is modelled by a Leslie matrix L. A *sustainable harvesting policy* is a method that allows a certain fraction of the population to be harvested without changing the size of the population. That is, if the current population vector is $\mathbf{x}$, and a fraction, h, of each age group is harvested, then sustainable harvesting requires that the population will still be $\mathbf{x}$ after one time period. (A time period is the length of each age group.)

In this model, beginning with a population vector $\mathbf{x}$, after one time period the population vector is $L\mathbf{x}$.
Harvesting a fraction h of this population leaves $L\mathbf{x} - hL\mathbf{x} = (1-h)L\mathbf{x}$.
For sustainability, we require

$$(1-h)L\mathbf{x} = \mathbf{x}.$$

When this equation is satisfied, h is called the *sustainable harvest ratio.*

(a) If λ_1 is the unique positive eigenvalue of L, show that the sustainable harvest ratio is $1 - 1/\lambda_1$.

(b) Find the sustainable harvest ratio corresponding to each of the Leslie matrices in Question **5**.

7. Let $M = \begin{bmatrix} \frac{5}{4} & 2 & 1 \\ \frac{2}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{bmatrix}$ be the Leslie matrix for an animal population with 3 age groups.

(a) Show that 2 is an eigenvalue of M, and find a corresponding eigenvector.

(b) If the population distribution is stable, and the total population is 34000, how many animals are in each age group?

Further exercises

8. Let

$$L = \begin{bmatrix} b_1 & b_2 & b_3 \\ s_1 & 0 & 0 \\ 0 & s_2 & 0 \end{bmatrix}$$

be the Leslie matrix for an animal population with 3 age groups.

(a) If λ is an eigenvalue of L show that

$$\lambda^3 = b_1\lambda^2 + s_1 b_2 \lambda + s_1 s_2 b_3.$$

(b) Verify that $\mathbf{u} = \begin{bmatrix} 1 \\ \dfrac{s_1}{\lambda} \\ \dfrac{s_1 s_2}{\lambda^2} \end{bmatrix}$ is an eigenvector of L corresponding to eigenvalue λ.

(c) The *net reproduction rate* for the population is defined to be

$$b_1 + s_1 b_2 + s_1 s_2 b_3.$$

(Note that this can be interpreted as the average number of female offspring born to a single female over her lifetime. Can you see why?)

Show that if $\lambda = 1$ the net reproduction rate is 1.

9. Let $P = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}$ be a stochastic matrix. (That is, each entry is positive, and the entries in each column add to 1.)

Prove that P has an eigenvalue of 1.

Answers to selected exercises

1. The corresponding eigenvalues are 1, 3 and 5.

2. (a) 5, −1, 4. (b) 3, 7, 1.

3. (a) Eigenvalues: −1, 1, 2.

Corresponding eigenvectors: $\begin{bmatrix} -\frac{t}{2} \\ -\frac{t}{2} \\ t \end{bmatrix}, \begin{bmatrix} -t \\ t \\ 0 \end{bmatrix}, \begin{bmatrix} t \\ t \\ t \end{bmatrix}$.

(b) Eigenvalues: −1, 3.

Corresponding eigenvectors: $\begin{bmatrix} -t \\ s \\ t \end{bmatrix}, \begin{bmatrix} t \\ \frac{3t}{2} \\ t \end{bmatrix}$.

4. (c) $\begin{bmatrix} \frac{24}{79} \\ \frac{28}{79} \\ \frac{27}{79} \end{bmatrix}$

5. (a) $\sqrt{2}, \begin{bmatrix} 3\sqrt{2} \\ 2 \end{bmatrix}$.

(b) $2, \begin{bmatrix} 8 \\ 3 \\ 1 \end{bmatrix}$.

6. (b) For 5(a): $h = 1 - \dfrac{1}{\sqrt{2}} \approx 0.29$. For 5(b): $h = \dfrac{1}{2}$.

7. (b) 24000 in the first age group, 8000 in the second age group, 2000 in the third age group.