The University of Sydney
School of Mathematics and Statistics

Tutorial 12 (Week 12)

MATH1014: Introduction to Linear Algebra — Semester 2, 2009

Preparatory questions (attempt before the tutorial)

1. Show that $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -19 \\ -4 \\ 1 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ are eigenvectors of the matrix $A = \begin{bmatrix} 2 & 4 & -3 \\ 0 & 3 & 0 \\ -1 & 5 & 4 \end{bmatrix}$. What are the corresponding eigenvalues?

Tutorial exercises

2. Write down the eigenvalues of each of the following matrices.

(a) $\begin{bmatrix} 5 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 0 & 0 \\ 6 & 7 & 0 \\ 9 & 2 & 1 \end{bmatrix}$

3. Find the eigenvalues, and corresponding eigenvectors of each of the following matrices:

(a) $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ (b) $B = \begin{bmatrix} 1 & 0 & 2 \\ 3 & -1 & 3 \\ 2 & 0 & 1 \end{bmatrix}$

4. Let $P = \begin{bmatrix} 0.2 & 0.3 & 0.4 \\ 0.6 & 0.1 & 0.4 \\ 0.2 & 0.6 & 0.2 \end{bmatrix}$ be the transition matrix of a Markov process.

(a) Verify that P has an eigenvalue of 1.

(b) Show that $\mathbf{u} = \begin{bmatrix} 24 \\ 28 \\ 27 \end{bmatrix}$ is a corresponding eigenvector.

(c) Find the steady state probability vector.

5. Every Leslie matrix has a unique positive eigenvalue, and a corresponding eigenvector with positive entries. Find the unique positive eigenvalue, and a corresponding eigenvector, for each of the following Leslie matrices.

(a) $L = \begin{bmatrix} 0 & 3 \\ \frac{2}{3} & 0 \end{bmatrix}$ (b) $M = \begin{bmatrix} 0 & 4 & 4 \\ \frac{3}{4} & 0 & 0 \\ 0 & \frac{2}{3} & 0 \end{bmatrix}$

6. Consider an animal population whose growth is modelled by a Leslie matrix L. A *sustainable harvesting policy* is a method that allows a certain fraction of the population to be harvested without changing the size of the population. That is, if the current population vector is $\mathbf{x}$, and a fraction, h, of each age group is harvested, then sustainable harvesting requires that the population will still be $\mathbf{x}$ after one time period. (A time period is the length of each age group.)

In this model, beginning with a population vector $\mathbf{x}$, after one time period the population vector is $L\mathbf{x}$.
Harvesting a fraction h of this population leaves $L\mathbf{x} - hL\mathbf{x} = (1-h)L\mathbf{x}$.
For sustainability, we require

$$(1-h)L\mathbf{x} = \mathbf{x}.$$

When this equation is satisfied, h is called the *sustainable harvest ratio.*

(a) If λ_1 is the unique positive eigenvalue of L, show that the sustainable harvest ratio is $1 - 1/\lambda_1$.

(b) Find the sustainable harvest ratio corresponding to each of the Leslie matrices in Question **5**.

7. Let $M = \begin{bmatrix} \frac{5}{4} & 2 & 1 \\ \frac{2}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{bmatrix}$ be the Leslie matrix for an animal population with 3 age groups.

(a) Show that 2 is an eigenvalue of M, and find a corresponding eigenvector.

(b) If the population distribution is stable, and the total population is 34000, how many animals are in each age group?

Further exercises

8. Let

$$L = \begin{bmatrix} b_1 & b_2 & b_3 \\ s_1 & 0 & 0 \\ 0 & s_2 & 0 \end{bmatrix}$$

be the Leslie matrix for an animal population with 3 age groups.

(a) If λ is an eigenvalue of L show that

$$\lambda^3 = b_1\lambda^2 + s_1 b_2 \lambda + s_1 s_2 b_3.$$

(b) Verify that $\mathbf{u} = \begin{bmatrix} 1 \\ \dfrac{s_1}{\lambda} \\ \dfrac{s_1 s_2}{\lambda^2} \end{bmatrix}$ is an eigenvector of L corresponding to eigenvalue λ.

(c) The *net reproduction rate* for the population is defined to be

$$b_1 + s_1 b_2 + s_1 s_2 b_3.$$

(Note that this can be interpreted as the average number of female offspring born to a single female over her lifetime. Can you see why?)

Show that if $\lambda = 1$ the net reproduction rate is 1.

9. Let $P = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}$ be a stochastic matrix. (That is, each entry is positive, and the entries in each column add to 1.)

Prove that P has an eigenvalue of 1.

Solutions

1. $A\,\mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} = \mathbf{v}_1$, so $\mathbf{v}_1$ is an eigenvector corresponding to eigenvalue 1.

$A\,\mathbf{v}_2 = \begin{bmatrix} -57 \\ -12 \\ 3 \end{bmatrix} = 3\,\mathbf{v}_2$, so $\mathbf{v}_2$ is an eigenvector corresponding to eigenvalue 3.

$A\,\mathbf{v}_3 = \begin{bmatrix} -5 \\ 0 \\ 5 \end{bmatrix} = 5\,\mathbf{v}_3$, so $\mathbf{v}_3$ is an eigenvector corresponding to eigenvalue 5.

2. The matrices are triangular, and hence the eigenvalues are the entries on the diagonal.

(a) 5, −1, 4. (b) 3, 7, 1.

3. (a) To find the eigenvalues, we solve the equation $\det(A - \lambda I) = |A - \lambda I| = 0$, for λ.

$$\begin{aligned}
\text{Now, } |A - \lambda I| &= \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} \\
&= -\lambda(1-\lambda)^2 - (1-\lambda) - (1-\lambda) \\
&= (1-\lambda)(\lambda^2 - \lambda - 2) = (1-\lambda)(\lambda-2)(\lambda+1).
\end{aligned}$$

So $|A - \lambda I| = 0$ when $\lambda = -1, 1, 2$ and hence the eigenvalues are $-1, 1, 2$.

To find the eigenvectors corresponding to $\lambda = -1$ we solve the system of equations $(A + I)\,\mathbf{x} = \mathbf{0}$.

The matrix $A + I = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, and this reduces to $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

Hence, the eigenvectors corresponding to the eigenvalue −1 are all vectors of the form $\begin{bmatrix} -\frac{t}{2} \\ -\frac{t}{2} \\ t \end{bmatrix}$, $t \in \mathbb{R}$, $t \neq 0$.

Another way of saying this is: the (-1)-eigenspace of A is the set of all vectors $\begin{bmatrix} -\frac{t}{2} \\ -\frac{t}{2} \\ t \end{bmatrix}$, $t \in \mathbb{R}$, $t \neq 0$.

To find the eigenvectors corresponding to $\lambda = 1$ we solve $(A - I)\,\mathbf{x} = \mathbf{0}$.

$$A - I = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}.$$

Hence, the eigenvectors are all vectors of the form $\begin{bmatrix} -t \\ t \\ 0 \end{bmatrix}$, $t \in \mathbb{R}$, $t \neq 0$.

(Or, the 1-eigenspace of A is the set of vectors $\begin{bmatrix} -t \\ t \\ 0 \end{bmatrix}$, $t \in \mathbb{R}$, $t \neq 0$.)

To find the eigenvectors corresponding to $\lambda = 2$ we solve $(A - 2I)\,\mathbf{x} = \mathbf{0}$.

$A - 2I = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & -2 \end{bmatrix}$, which reduces to $\begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

Hence, the eigenvectors are all vectors of the form $\begin{bmatrix} t \\ t \\ t \end{bmatrix}$, $t \in \mathbb{R}$, $t \neq 0$.

(The 2-eigenspace of A is the set of vectors $\begin{bmatrix} t \\ t \\ t \end{bmatrix}$, $t \in \mathbb{R}$, $t \neq 0$.)

(b) To find the eigenvalues, we solve the equation $|B - \lambda I| = 0$, for λ.

$$\begin{aligned} |B - \lambda I| &= \begin{vmatrix} 1-\lambda & 0 & 2 \\ 3 & -1-\lambda & 3 \\ 2 & 0 & 1-\lambda \end{vmatrix} \\ &= (1-\lambda)^2(-1-\lambda) - 4(-1-\lambda) \\ &= -(1+\lambda)(\lambda^2 - 2\lambda - 3) = -(1+\lambda)^2(\lambda - 3) \end{aligned}$$

So $|B - \lambda I| = 0$ when $\lambda = -1, 3$ and hence the eigenvalues are -1, 3.

To find the eigenvectors corresponding to $\lambda = -1$ we solve $(B + I)\,\mathbf{x} = \mathbf{0}$.

$B + I = \begin{bmatrix} 2 & 0 & 2 \\ 3 & 0 & 3 \\ 2 & 0 & 2 \end{bmatrix}$, which reduces to $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Note that there are *two* parameters in the solution. The eigenvectors are all vectors of the form $\mathbf{x} = \begin{bmatrix} -t \\ s \\ t \end{bmatrix}$, $s, t \in \mathbb{R}$, $\mathbf{x} \neq 0$.

To find the eigenvectors corresponding to $\lambda = 3$ we solve $(B - 3I)\,\mathbf{x} = \mathbf{0}$.

$B - 3I = \begin{bmatrix} -2 & 0 & 2 \\ 3 & -4 & 3 \\ 2 & 0 & -2 \end{bmatrix}$, which reduces to $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -\frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$.

Hence, the eigenvectors are all vectors of the form $\begin{bmatrix} t \\ \frac{3t}{2} \\ t \end{bmatrix}$, $t \in \mathbb{R}$, $t \neq 0$.

4. (a) The simplest way to verify that 1 is an eigenvalue of P is to show that the determinant of $P - I$ is zero.

Using expansion by the first row:

$$\begin{aligned}|P - I| &= \begin{vmatrix}-0.8 & 0.3 & 0.4\\ 0.6 & -0.9 & 0.4\\ 0.2 & 0.6 & -0.8\end{vmatrix}\\ &= -0.8\begin{vmatrix}-0.9 & 0.4\\ 0.6 & -0.8\end{vmatrix} - 0.3\begin{vmatrix}0.6 & 0.4\\ 0.2 & -0.8\end{vmatrix} + 0.4\begin{vmatrix}0.6 & -0.9\\ 0.2 & 0.6\end{vmatrix}\\ &= 0.\end{aligned}$$

This verifies that P has an eigenvalue of 1.

[*Alternate method*:

Apply, in turn, the row operations $R3 \to R3 + R1$ and $R3 \to R3 + R2$ to the matrix $P - I$:

$$P - I = \begin{bmatrix}-0.8 & 0.3 & 0.4\\ 0.6 & -0.9 & 0.4\\ 0.2 & 0.6 & -0.8\end{bmatrix} \to \begin{bmatrix}-0.8 & 0.3 & 0.4\\ 0.6 & -0.9 & 0.4\\ 0 & 0 & 0\end{bmatrix}.$$

The row operations do not alter the value of the determinant, and the determinant of the reduced matrix is clearly zero. Hence, $|P - I| = 0$.]

(b) $$P\mathbf{u} = \begin{bmatrix}0.2 & 0.3 & 0.4\\ 0.6 & 0.1 & 0.4\\ 0.2 & 0.6 & 0.2\end{bmatrix}\begin{bmatrix}24\\ 28\\ 27\end{bmatrix} = \begin{bmatrix}24\\ 28\\ 27\end{bmatrix} = \mathbf{u},$$

so $\mathbf{u}$ is an eigenvector corresponding to the eigenvalue 1.

(c) The steady state probability vector is a vector $\mathbf{v}$ such that $P\mathbf{v} = \mathbf{v}$, and the entries of $\mathbf{v}$ add to 1. That is, $\mathbf{v}$ is an eigenvector of P corresponding to the eigenvalue 1, with entries that add to 1. Now, any vector $\begin{bmatrix}24t\\ 28t\\ 27t\end{bmatrix}$ (where t is any real number) is an eigenvector corresponding to the eigenvalue 1, and if the entries are to add to 1, we require $t = 1/79$.

Hence, the steady state probability vector is $\begin{bmatrix}\frac{24}{79}\\ \frac{28}{79}\\ \frac{27}{79}\end{bmatrix}$.

5. (a) $|L - \lambda I| = \lambda^2 - 2 = 0$ when $\lambda = \pm\sqrt{2}$. So the positive eigenvalue of L is $\sqrt{2}$.

$$L - \sqrt{2}I = \begin{bmatrix}-\sqrt{2} & 3\\ \frac{2}{3} & -\sqrt{2}\end{bmatrix} \xrightarrow{R1 \to R1 \div (-\sqrt{2})} \begin{bmatrix}1 & -\frac{3\sqrt{2}}{2}\\ \frac{2}{3} & -\sqrt{2}\end{bmatrix} \xrightarrow{R2 \to R2 - \frac{2}{3}R1} \begin{bmatrix}1 & -\frac{3\sqrt{2}}{2}\\ 0 & 0\end{bmatrix}.$$

So a corresponding eigenvector is $\begin{bmatrix}3\sqrt{2}\\ 2\end{bmatrix}$.

(b) $$|M - \lambda I| = \begin{vmatrix}-\lambda & 4 & 4\\ \frac{3}{4} & -\lambda & 0\\ 0 & \frac{2}{3} & -\lambda\end{vmatrix} = -\lambda^3 + 3\lambda + 2.$$

By observation, we see that $-\lambda^3 + 3\lambda + 2 = 0$ when $\lambda = 2$. So 2 is the positive eigenvalue of M.

Now, $M - 2I = \begin{bmatrix} -2 & 4 & 4 \\ \frac{3}{4} & -2 & 0 \\ 0 & \frac{2}{3} & -2 \end{bmatrix}$, which reduces to $\begin{bmatrix} 1 & -2 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$.

A corresponding eigenvector is $\begin{bmatrix} 8 \\ 3 \\ 1 \end{bmatrix}$.

6. (a) If λ_1 is the unique positive eigenvalue, then $L\mathbf{x} = \lambda_1 \mathbf{x}$.
For sustainability, we require that $(1-h)L\mathbf{x} = (1-h)\lambda_1 \mathbf{x} = \mathbf{x}$.
Hence, $(1-h)\lambda_1 = 1$, or $1 - h = 1/\lambda_1$ (noting that $\lambda_1 \neq 0$) and therefore $h = 1 - 1/\lambda_1$.

(b) For the population with Leslie matrix L, $h = 1 - \dfrac{1}{\sqrt{2}} \approx 0.29$. (So 29% of the population may be harvested without depleting the population.)
For the population with Leslie matrix M, $h = 1 - \dfrac{1}{2} = \dfrac{1}{2}$. (So half the population may be harvested without depleting the population.)

7. (a) Similarly to question 4(a), we show that $|M - 2I| = 0$ in order to verify that 2 is an eigenvalue.

$|M - 2I| = \begin{vmatrix} -\frac{3}{4} & 2 & 1 \\ \frac{2}{3} & -2 & 0 \\ 0 & \frac{1}{2} & -2 \end{vmatrix}$, and expanding by the first column we have:

$$|M - 2I| - \frac{3}{4} \times 4 - \frac{2}{3} \times (-4 - \frac{1}{2}) = 0.$$

[*An alternate method*:

$$|M - \lambda I| = \begin{vmatrix} \frac{5}{4} - \lambda & 2 & 1 \\ \frac{2}{3} & -\lambda & 0 \\ 0 & \frac{1}{2} & -\lambda \end{vmatrix} = -\lambda^3 + \frac{5\lambda^2}{4} + \frac{4\lambda}{3} + \frac{1}{3} = 0 \text{ when } \lambda = 2.]$$

Now, the matrix $M - 2I$ reduces to $\begin{bmatrix} 1 & -\frac{8}{3} & -\frac{4}{3} \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{bmatrix}$, and so a corresponding eigenvector is $\begin{bmatrix} 12 \\ 4 \\ 1 \end{bmatrix}$.

(b) From the eigenvector $\begin{bmatrix} 12 \\ 4 \\ 1 \end{bmatrix}$, we see that when the population distribution is stable, it is divided between the age groups in the ratio $12 : 4 : 1$. So if the total population is 34000, there are 24000 animals in the first age group, 8000 in the second age group and 2000 in the third age group.

8. (a) If λ is an eigenvalue of L, then $|L - \lambda I| = 0$.

$$\begin{aligned}\text{Now, } |L - \lambda I| &= \begin{bmatrix} b_1 - \lambda & b_2 & b_3 \\ s_1 & -\lambda & 0 \\ 0 & s_2 & -\lambda \end{bmatrix} \\ &= (b_1 - \lambda)\lambda^2 - s_1(-b_2\lambda - s_2 b_3) \\ &= -\lambda^3 + b_1\lambda^2 + s_1 b_2 \lambda + s_1 s_2 b_3.\end{aligned}$$

So when $|L - \lambda I| = 0$ we have $\lambda^3 = b_1\lambda^2 + s_1 b_2\lambda + s_1 s_2 b_3$.

(b) $$L\,\mathbf{u} = \begin{bmatrix} b_1 & b_2 & b_3 \\ s_1 & 0 & 0 \\ 0 & s_2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \frac{s_1}{\lambda} \\ \frac{s_1 s_2}{\lambda^2} \end{bmatrix} = \begin{bmatrix} b_1 + \frac{s_1 b_2}{\lambda} + \frac{s_1 s_2 b_3}{\lambda^2} \\ s_1 \\ \frac{s_1 s_2}{\lambda} \end{bmatrix}.$$

$$\begin{aligned}\text{Now, } b_1 + \frac{s_1 b_2}{\lambda} + \frac{s_1 s_2 b_3}{\lambda^2} &= \frac{b_1\lambda^2 + s_1 b_2 \lambda + s_1 s_2 b_3}{\lambda^2} \\ &= \frac{\lambda^3}{\lambda^2} \quad \text{(from part (a))} \\ &= \lambda.\end{aligned}$$

So $L\,\mathbf{u} = \begin{bmatrix} \lambda \\ s_1 \\ \frac{s_1 s_2}{\lambda} \end{bmatrix} = \lambda\,\mathbf{u}$, thereby verifying that $\mathbf{u}$ is an eigenvector.

(c) In part (a) we have $\lambda^3 = b_1\lambda^2 + s_1 b_2\lambda + s_1 s_2 b_3$.
When $\lambda = 1$ this equation becomes $b_1 + s_1 b_2 + s_1 s_2 b_3 = 1$.

9. We can prove that P has an eigenvalue of 1 by showing that $P - I = 0$.

$P - I = \begin{bmatrix} p_{11} - 1 & p_{12} & p_{13} \\ p_{21} & p_{22} - 1 & p_{23} \\ p_{31} & p_{32} & p_{33} - 1 \end{bmatrix}$, and applying, in turn, the row operations $R3 \to R3 + R1$ and $R3 \to R3 + R2$ we obtain

$$\begin{bmatrix} p_{11} - 1 & p_{12} & p_{13} \\ p_{21} & p_{22} - 1 & p_{23} \\ p_{11} + p_{21} + p_{31} - 1 & p_{12} + p_{22} + p_{32} - 1 & p_{13} + p_{23} + p_{33} - 1 \end{bmatrix}.$$

But P is a stochastic matrix, and so

$$p_{11} + p_{21} + p_{31} = 1, \;\; p_{12} + p_{22} + p_{32} = 1, \;\; p_{13} + p_{23} + p_{33} = 1.$$

Hence, $P - I$ reduces to

$$\begin{bmatrix} p_{11} - 1 & p_{12} & p_{13} \\ p_{21} & p_{22} - 1 & p_{23} \\ 0 & 0 & 0 \end{bmatrix}.$$

The row operations do not alter the value of the determinant, and the determinant of the reduced matrix is clearly zero. Hence, $|P - I| = 0$.